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Iterative Method with Inertia for Variational Inequalities on Hadamard Manifolds with Lower Bounded Curvature

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Abstract. In this paper, we are concerned with solving variational inequalities on Hadamard manifolds, the curvature of which is bounded from below. The underlying vector field is assumed to be continuous and pseudomonotone. By combining the hyperplane projection method and the inertial extrapolation technique, a Halpern-type method is proposed. Under some mild assumptions, global convergence of the proposed algorithm is established. Numerical experiments are reported to show the efficiency of the proposed algorithm.

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Key words: Variational inequality, pseudomonotone vector field, Hadamard manifold, hyperplane projection method, inertial term.

1. Introduction

In the past few decades, various optimization problems on Riemannian manifolds have been studied [1, 3, 7, 11, 12, 14, 16, 18, 21, 22, 26, 30–33, 35–37, 41–43]. Among these research problems, variational inequality problems on Riemannian manifolds have received much attention. Specially, a complete and simply connected Riemannian manifold with nonpositive curvature is called the Hadamard manifold. Variational inequality problems for univalued vector fields on Hadamard manifolds was first introduced and investigated by Németh [22]. The existence and uniqueness results were established. Li *et al.* [19] introduced variational inequality problems for univalued vector fields on general Riemannian manifolds, the existence results of solution for problems defined on locally convex subsets

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with weak pole interior points were established. For set-valued vector fields on general Riemannian manifolds, the existence of the solutions and the convexity of the solution set of related variational inequality problems are investigated by Li *et al.* [17]. In addition, nonsmooth variational inequalities on Hadamard manifolds were also studied by Ansaria *et al*. [4]. Some existence results in terms of a bifunction in the setting of Hadamard manifolds were given.

By the Cartan-Hadamard theorem, any Hadamard manifold is diffeomorphic to a Euclidean space. Thus variational inequality problems on a Hadamard manifold can be reformulated as some equivalent problem in a Euclidean space by using the underlying diffeomorphic map. However, the monotonicity/pseudomonotonicity of the underlying vector field may be destroyed. Conversely, by endowing some appropriate Riemannian metric on a subset of a Euclidean space, this set may become a Hadamard manifold, and some nonmonotone problems on the original set may become monotone problems with respect to the endowed Riemannian metric [10]. Thus how to use the geometric structure of Hadamard manifolds directly to devise efficient algorithms becomes an important research topic.

In the past few years, some iterative methods have been proposed for solving variational inequalities on Riemannian manifolds. Li *et al.* [17] proposed a proximal point algorithm for variational inequality problems for set-valued monotone vector fields on general Riemannian manifolds. Tang *et al.* [25] constructed a variant of Korpelevich's method for variational inequality problems for univalued pseudomonotone vector fields on Hadamard manifolds. Tang *et al.* [27] proposed a modified projection-type method for variational inequality problems for univalued pseudomonotone vector fields on Hadamard manifolds. Batista *et al.* [5,6] introduced an inexact proximal point algorithm and an extragradient-type algorithm for variational inequality for set-valued monotone vector fields on Hadamard manifolds. Ansaria *et al.* [2] considered a proximal point algorithm for inclusion problems on Hadamard manifolds and its application to univalued monotone variational inequality problem.

In this paper, we are concerned with the construction of an efficient iterative method for solving variational inequality problems for univalued pseudomonotone vector field on Hadamard manifolds. Let $\mathscr{H}$ be a Hadamard manifold. Without abuse of notation, we use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to denote the inner products and norms on different tangent spaces of $\mathscr{H}$ by its Riemannian metric. A tangent vector field $V:\mathscr{H}\to T\mathscr{H}$ is called pseudomonotone if it holds

$$\left\langle \mathrm{Exp}_p^{-1}q, V(p)\right\rangle \geq 0 \implies \left\langle \mathrm{Exp}_q^{-1}p, V(q)\right\rangle \leq 0 \quad \text{for all} \quad p,q\in\mathscr{H}, \tag{1.1}$$

where $\mathrm{Exp}_p^{-1}:\mathscr{H}\to T_p\mathscr{H}$ denotes the inverse of the exponential map $\mathrm{Exp}_p:T_p\mathscr{H}\to\mathscr{H}$. A subset $C\subset\mathscr{H}$ is said to be convex if for any two points $p,q\in C$, the Riemannian geodesic connecting p and q is contained in C; that is, if $\gamma:[a,b]\to C$ is a Riemannian geodesic with $\gamma(a)=p$ and $\gamma(b)=q$, then $\gamma((1-t)a+tb)\in C$ for any $t\in[0,1]$. Let C be a closed and convex subset of $\mathscr{H}$ and $V:\mathscr{H}\to T\mathscr{H}$ be a pseudomonotone vector field. The variational inequality problem for V and C is to find a point $p\in C$ such that

$$\left\langle V(p), \mathrm{Exp}_p^{-1}q\right\rangle \geq 0 \quad \text{for all} \quad q\in C. \tag{1.2}$$