East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-085.070723

Vol. **14**, No. 4, pp. 731-768
November 2024

Error Analysis of BDF-Galerkin FEMs for Thermally Coupled Incompressible MHD with Temperature Dependent Parameters

Shuaijun Liu, Pengzhan Huang* and Yinnian He

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, P.R. China.

Received 17 March 2023; Accepted (in revised version) 7 July 2023.

Abstract. In this paper, we consider the electromagnetically and thermally driven flow which is modeled by evolutionary magnetohydrodynamic equations and heat equation coupled through generalized Boussinesq approximation with temperature-dependent coefficients. Based on a third-order backward differential formula for temporal discretization, mixed finite element approximation for spatial discretization and extrapolated treatments in linearization for nonlinear terms, a linearized backward differentiation formula type scheme for the considered equations is proposed and analysed. Optimal L^2-error estimates for the proposed fully discretized scheme are obtained by the temporal-spatial error splitting technique. Numerical examples are presented to check the accuracy and efficiency of the scheme.

AMS subject classifications: 65M12, 65M60

Key words: Thermally coupled magnetohydrodynamic, Boussinesq approximation, temperature dependent coefficient, linearized BDF scheme, convergence.

1. Introduction

The hydrodynamical behavior of conducting fluids subject to external magnetic fields can be well described by magnetohydrodynamic (MHD) equations which are governed by a combination of Navier-Stokes equations and Maxwell's equations. Due to wide applications of MHD systems in astronomy, geophysics and engineering, such as metallurgical engineering, contactless electromagnetic stirring, the design of cooling systems with liquid metals for a nuclear reactor, damping convective flow in metal-like melt and so on [6, 37], it is important to find accurate effective numerical methods for their solution.

Besides, the buoyancy effect can not be ignored in the momentum equation since the fluid will produce temperature difference during conduction. Therefore, MHD systems are

*Corresponding author. *Email addresses:* SJ_Liu123@163.com (S. Liu), hpzh@xju.edu.cn (P. Huang), heyn@mail.xjtu.edu.cn (Y. He)

usually coupled with the heat equation. Furthermore, in many applications, the properties of fluids and materials in MHD system may not be constant. For example, viscosity, thermal diffusivity and magnetic diffusivity can be strongly dependent on temperature, which will lead to the original system becoming a stronger nonlinearity and coupling system [5,14,48].

Let Ω be a connected bounded open set in $R^d, d=2$ or 3, either convex or having a $C^{1,1}$ boundary $\partial\Omega$ and $[0,T]$ an interval of R. In this paper, we deal with a time-dependent thermally coupled incompressible MHD flow. More exactly, we consider the following incompressible Navier-Stokes and Maxwell equations coupled with the heat equation by the well-known Boussinesq approximation

$$\begin{aligned} &\partial_t\mathbf{u}-\nabla\cdot\big(\nu(\theta)\nabla\mathbf{u}\big)+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p+s\mathbf{b}\times\mathrm{curl}\mathbf{b}-\beta(\theta)\theta\mathbf{j}=\mathbf{f} && \text{in } \Omega\times(0,T],\\ &\partial_t\mathbf{b}+\mathrm{curl}\big(\mu(\theta)\mathrm{curl}\mathbf{b}\big)-\mathrm{curl}(\mathbf{u}\times\mathbf{b})=\mathbf{0} && \text{in } \Omega\times(0,T],\\ &\partial_t\theta-\nabla\cdot\big(\kappa(\theta)\nabla\theta\big)+(\mathbf{u}\cdot\nabla)\theta=\psi && \text{in } \Omega\times(0,T],\\ &\mathrm{div}\,\mathbf{u}=0,\quad \mathrm{div}\,\mathbf{b}=0 && \text{in } \Omega\times(0,T], \end{aligned} \tag{1.1}$$

where the fluid viscous diffusivity ν, the magnetic diffusivity μ, the thermal conductivity κ and the thermal expansion coefficient β depend on the temperature. The unknowns are velocity field $\mathbf{u}$, temperature θ, pressure p, and magnetic field $\mathbf{b}$. Functions $\mathbf{f}$ and ψ are respectively a known body force and a heat source, $\nu(\theta)$ denotes the fluid viscous diffusivity, $\kappa(\theta)$ the thermal conductivity, $\beta(\theta)$ the thermal expansion coefficient, and $\mu(\theta):=1/\eta_0\delta(\theta)$ the magnetic diffusivity, where η_0 and δ denote the magnetic permeability and the electrical conductivity. Besides, $\mathbf{j}$ is the unit vector in the direction opposite to the gravitation, and $s:=1/\eta_0\rho_0$ the coupling coefficient, where ρ_0 is the reference density.

The system (1.1) is considered in conjunction with the following initial values and boundary conditions:

$$\begin{aligned} &\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0, \mathbf{b}(\mathbf{x},0)=\mathbf{b}_0, \theta(\mathbf{x},0)=\theta_0 && \text{for all}\quad \mathbf{x}\in\Omega,\\ &\mathbf{u}|_{\partial\Omega_T}=0, && \text{(no-slip condition)},\\ &\mathbf{b}\cdot\mathbf{n}|_{\partial\Omega_T}=0,\quad \mathrm{curl}\mathbf{b}\times\mathbf{n}|_{\partial\Omega_T}=\mathbf{0}, && \text{(perfectly conducting wall)},\\ &\theta|_{\partial\Omega_T}=0, \end{aligned} \tag{1.2}$$

where $\mathbf{n}$ is the outer unit normal of $\partial\Omega$, $\partial\Omega_T=\partial\Omega\times(0,T]$ and the initial magnetic induction $\mathbf{b}_0$ satisfies $\mathrm{div}\,\mathbf{b}_0=0$.

It is worth noting that in recent years, various analytical investigations and have been carried out and numerous efficient numerical methods for solving MHD systems have been developed. For stationary MHD problems, error estimates of finite element methods are given in [9, 12, 15, 16, 43, 46, 52]. On the other hand, for time-dependent MHD systems, the convergence analysis and error estimates of first- and second-order-in-time for fully discrete finite element methods are respectively established in [11, 17, 28, 30, 38, 54, 56] and [8,19,41,47,55,57]. For example, a fully discrete linearized H^1-conforming Lagrange finite element method for a 2D vector potential MHD model is proposed in [25]. Wang *et al.* [47] introduced a numerical scheme based on the modified Crank-Nicolson finite element projection method and obtained optimal error estimates for MHD equations. Kanbar