

Pointwise Error Estimates of $L1$ Method for Multi-Singularity Problems Arising in Delay Fractional Equations

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Abstract. Error estimates of $L1$ scheme for delay fractional equations are derived by discrete Laplace transform method. Theoretical result shows that the convergence order is $\min\{(k+1)\alpha, 1\}$ at $(k\tau)^+$, where $k \in \mathbb{N}$, τ is delay factor, $\alpha \in (0, 1)$ is the order of Caputo fractional derivative. At the points without derivative discontinuities, first order convergence is achieved. The uniqueness of the inverse problem, the reaction coefficient, and the delay factor are established by employing asymptotic expansions and the monotonicity of the Mittag-Leffler function. An inversion algorithm based on the Tikhonov regularization method is given.

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1. Introduction

We start with a simple popular delay fractional model — viz.

$${}_0^C D_t^\alpha u(t) + \lambda u(t) + b(t)u(t - \tau) = f(t), \quad t \in (0, T], \quad (1.1)$$

$$u(t) = \phi(t), \quad t \in [-\tau, 0], \quad (1.2)$$

where λ is a positive constant, $b(t)$ a smooth time-dependent function, $\tau > 0$ a constant time delay, $T = K\tau$, $K \in \mathbb{Z}^+$, and

$${}_a^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(s)}{(t-s)^\alpha} ds, \quad a \geq 0, \quad \alpha \in (0, 1).$$

This problem attracted considerable interest in recent years. If $\lambda = 0$ and $f(t) = 0$, the stability regions of the model are given in [5]. Furthermore, the finite time stability of the

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problem has been proven by using a delayed Mittag-Leffler type matrix function — cf. [16]. If $b(t)$ is a time-independent function and $f(t) = 0$, the stability and the asymptotics of the solutions are derived in [4]. For generalised nonlinear systems, the existence, uniqueness, exponential boundedness, and convergence of solutions have been studied in [31, 32].

As theoretical foundations gradually improve, numerical methods have begun to receive attention. The existing numerical analysis is usually based on the assumption that the solutions are smooth [8, 34–36]. However, an initial singularity may propagate at integer multiples of the fixed delay even in the case of smooth source functions $f(t)$ [2, 23, 29]. Note that global error estimates are recently established under singularity conditions [1, 2]. There are also studies concerning fractional equations and weak singularities [3, 6, 7, 12–15, 17–19, 21, 22, 26, 27, 30, 37]. Hence, it is natural to pay attention to pointwise error estimates in the following stage. It is motivation behind the current work. Then, the proposed $L1$ scheme is applied in numerical algorithm for inverse problem.

The main contributions of this work are as follows:

1. Assuming the regularity of the initial function and the source term, we propose pointwise error estimates of the $L1$ scheme for delay fractional equations.
2. We show the unique solvability of the inverse problem involving a fractional order, the reaction coefficient and the delay factor, and introduce an inversion algorithm based on the Tikhonov regularization method.

The structure of the paper is as follows. In Section 2, an $L1$ scheme in the form of discrete Laplace transform is proposed. In Section 3, pointwise error estimates are derived under the multi-singularity conditions. In Section 4, simultaneous inversion of multi-parameters is considered, and analysis of uniqueness and inversion algorithm are obtained. Numerical tests are carried out in Section 5.

2. $L1$ Scheme in Form of Discrete Laplace Transform

Let

$$v(t) = u(t) - u(0), \quad g(t) = f(t) - b(t)u(t - \tau) - f(0) + b(0)u(-\tau).$$

Then the Eqs. (1.1)-(1.2) take the form

$${}_0^C D_t^\alpha v(t) + \lambda v(t) = g(t) - \lambda u(0) + f(0) - b(0)u(-\tau), \quad t \in (0, T], \quad (2.1)$$

$$v(t) = \phi(t) - u(0), \quad t \in [-\tau, 0]. \quad (2.2)$$

Applying the Laplace transform to (2.1)-(2.2) and letting

$$\varphi = -\lambda u(0) + f(0) - b(0)u(-\tau),$$

one has

$$z^\alpha \hat{v}(z) + \lambda \hat{v}(z) = z^{-1} \varphi + \hat{g}(z),$$