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The Regularized Global GMERR Method for Solving Large-Scale Linear Discrete Ill-Posed Problems

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Abstract. For the large-scale linear discrete ill-posed problems with multiple right-hand sides, the global Krylov subspace iterative methods have received a lot of attention. In this paper, we analyze the regularizing properties of the global generalized minimum error method (GMERR), and develop a regularized global GMERR method for solving linear discrete ill-posed problems with multiple right-hand sides. The efficiency of the proposed method is confirmed by the numerical experiments on test matrices.

AMS subject classifications: 65F10, 65F22
Key words: Linear discrete ill-posed problems, multiple right-hand sides, global GMERR method, regularizing properties.

1. Introduction

We are interested in approximate solutions of the large-scale linear discrete ill-posed problems
$$Ax^{(j)} = b^{(j)}, \quad j = 1, 2, \ldots, s, \tag{1.1}$$
which have the same coefficient matrix $A \in \mathbf{R}^{n\times n}$ and different right-hand sides $b^{(j)}$, $j = 1, 2, \ldots, s$. These problems arise in real-world applications, including electromagnetic wave scattering problem [45], pattern classification [12], image restoration [36], and dimensionality reduction [52] and so on. The difficulty in solving the problems is that the coefficient matrix A is nonsingular, but ill-conditioned with its singular values decaying to zero with increasing index without a noticeable gap and the right-hand vectors $b^{(j)} = \hat{b}^{(j)} + e^{(j)} \in \mathbf{R}^n$, $j = 1, 2, \ldots, s$ are assumed to be contaminated by unknown error-free vectors $\hat{b}^{(j)}$ and unknown measurement errors or noises $e^{(j)}$. Writing
$$X = \left[x^{(1)}, x^{(2)}, \ldots, x^{(s)}\right], \quad B = \left[b^{(1)}, b^{(2)}, \ldots, b^{(s)}\right],$$
$$\hat{B} = \left[\hat{b}^{(1)}, \hat{b}^{(2)}, \ldots, \hat{b}^{(s)}\right], \quad E = \left[e^{(1)}, e^{(2)}, \ldots, e^{(s)}\right],$$
we refer to E as the error or noise matrix. Then $B = \hat{B} + E$, and (1.1) can be written as
$$AX = B. \tag{1.2}$$

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The unavailable linear system of equations

$$AX = \hat{B} \tag{1.3}$$

is assumed to be consistent — i.e. the right-hand side $\hat{B}$ is in the range of A. However, it is possible that the available error-contaminated right-hand side B is outside of the range of A. In fact, we are interested in the minimum Frobenius norm solution of (1.3), which will be denoted by $\hat{X} \in \mathbf{R}^{n\times s}$.

Let $\|E\|_F$ denote the Frobenius norm of the noise matrix E, and the bound of the norm of the noise matrix E be $\delta > 0$, i.e.

$$\|E\|_F \leq \delta. \tag{1.4}$$

We consider how to determine a meaningful approximation of $\hat{X}$ by computing a proper approximate solution of (1.2).

If the coefficient matrix A is well-conditioned, the linear system (1.2) may be solved by direct or iterative methods [2, 18], including matrix splitting iteration methods [2] and Krylov subspace methods [1, 18]. However, since the matrix splitting iteration methods cannot be directly applied to large-scale linear discrete ill-posed problems, more attention has been paid to Krylov subspace methods. Due to the ill-conditioning of the matrix A and the presence of a noise E in B, the native solution to problem (1.2) does not furnish a useful approximation of the true solution to problem (1.3). Therefore, regularization methods must be used to extract as good an approximation to the true solution as possible. Two of the most popular regularization methods are Tikhonov regularization method [37, 47] and truncated singular value decomposition regularization method (TSVD) [20–22], which are either computationally unfeasible or extremely time-consuming for large-scale problems. So iterative regularization has received considerable attention.

There are a number of Krylov subspace methods for solving large-scale linear discrete ill-posed problems with a single right-hand side

$$Ax = b, \quad b \in \mathbf{R}^n. \tag{1.5}$$

When A is symmetric positive definite, Gilyazov [17] and Hanke [19] developed the conjugate gradient (CG) method for solving the ill-posed problem (1.5). Plato [42] analyzed the regularizing properties of CG. Using the Lanczos tridiagonalization, Paige and Saunders [41] developed an MINRES method for solving the linear systems with symmetric indefinite matrix A. Hanke [19], Kilmer and Stewart [34], Jensen and Hansen [29], Huang and Jia [26] analyzed regularizing effects of MINRES and showed its semi-convergence. When A is nonsymmetric, Björck [6] presented the CGLS algorithm, which implicitly applies CG to the normal equation $A^TAx = A^Tb$. Hanke [19] studied the regularizing properties of CGLS. Based on the Lanczos bidiagonalization, Paige and Saunders [40] proposed the LSQR algorithm, which is mathematically equivalent to CGLS. The regularizing effects of the LSQR algorithm are analyzed in [25]. Fong and Saunders [14] developed an LSMR algorithm, which is equivalent to the MINRES method applied to the normal equation $A^TAx = A^Tb$. Jia [30] analyzed the regularization properties of the LSMR algorithm. The GMRES method [44] is a popular iterative method for solving large linear