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An ODE Approach to Multiple Choice Polynomial Programming

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Abstract. We propose an ODE approach to solving multiple choice polynomial programming (MCPP) after assuming that the optimum point can be approximated by the expected value of so-called thermal equilibrium as usually did in simulated annealing. The explicit form of the feasible region and the affine property of the objective function are both fully exploited in transforming an MCPP problem into an ODE system. We also show theoretically that a local optimum of the former can be obtained from an equilibrium point of the latter. Numerical experiments on two typical combinatorial problems, MAX-k-CUT and the calculation of star discrepancy, demonstrate the validity of the ODE approach, and the resulting approximate solutions are of comparable quality to those obtained by the state-of-the-art heuristic algorithms but with much less cost. When compared with the numerical results obtained by using Gurobi to solve MCPP directly, our ODE approach is able to produce approximate solutions of better quality in most instances. This paper also serves as the first attempt to use a continuous algorithm for approximating the star discrepancy.

AMS subject classifications: 90C59, 35Q90, 90C09, 90C23
Key words: Pseudo-Boolean optimization, multiple choice constraint, continuous approach, MAX-CUT, star discrepancy.

1. Introduction

We consider the following pseudo-Boolean optimization problem:

$$
\begin{aligned}
\min_{x\in\{0,1\}^n} \quad & f(x), \\
\text{s.t.} \quad & \sum_{i\in I_j} x_i = 1, \quad j = 1,2,\ldots,m,
\end{aligned} \tag{1.1}
$$

where f is a polynomial function, x is an n-dimensional Boolean vector, the indices $[n] := \{1,2,\ldots,n\}$ are divided into m disjoint subsets $I_1, I_2, \ldots, I_m$, and the cardinality of each I_j, denoted by $d_j := |I_j|$, must be greater than 1. Then x is accordingly divided into m vectors

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$x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ where each $x^{(j)}$ picks out all the entries in I_j of x. The constraints mean that, for each j, there exists exact one element that equals to 1 in the subvector $x^{(j)}$, implying that only one is determined from d_j choices and thus m items chosen from n choices in total. A group of entries of x in a single I_j represents a decision out of finite choices. More precisely, we use multiple choice polynomial programming to call the problem (1.1), which is capable of dealing with various problems in diverse disciplines [2], for example, the MAX-k-CUT [20], star discrepancy [8] problems and SAT [17]. It should be pointed out that studies on integer linear programming problems with multiple choice constraints, termed the multiple choice programming (MCP), can date back to [11]. Since then several methods have been developed — cf. [27], but most of them are not designed for nonlinear objective functions. Although MCPP can be transformed into MCP by defining new variables to represent monomials, an exploration in this direction will not be presented here.

As a typical 0-1 programming problem, MCPP can also be treated with standard mixed integer nonlinear programming solvers, such as Gurobi [10], IBM-CPLEX [14], and SCIP [1], but few of them are designed for general nonlinear programming. For example, to apply Gurobi, one has to reformulate the nonlinear terms in MCPP instances into linear and/or quadratic forms by defining new variables and new constraints, thereby greatly increasing the problem size. This defect becomes even severer in calculating the star discrepancy since the degree of corresponding objective function is nothing but the dimension of underlying space and is usually much greater than 1 (see Eq. (6.13) in Section 6).

Alternatively, continuous approaches to discrete problems has attracted more attention since the Hopfield network — cf. [13], an early ODE approach, was proposed for the 0-1 quadratic programming in 1980s. The Hopfield network has been extended to other combinatorial optimization problems [15,28], where many useful mathematical techniques in dynamical system have been introduced. In the most recent work [22], an ODE approach was proposed through a quartic penalty approximation of the Boolean polynomial program. In line with this, here we propose to use the solutions of the following ODE system:

$$\begin{aligned} &\frac{\mathrm{d}y_i^{(j)}}{\mathrm{d}t} = -y_i^{(j)} + \sigma_i\left(-\Phi^{(j)}(y); 1/T\right), \quad j \in [m], \quad i \in [d_j], \\ &y(0) = y_0 \in [0,1]^n \end{aligned} \tag{1.2}$$

to approximate the solutions of MCPP, where the time-dependent vector $y(t) : [0, +\infty) \to \mathbb{R}^n$ is divided as x in Eq. (1.1) does, the initial data y_0 is required to satisfy the continuous multiple choice constraint: $\sum_{i=1}^{d_j} (y_0)_i^{(j)} = 1$ for all $j \in [m]$, $\Phi^{(j)}$ denotes the partial derivative of f with respect to $x^{(j)}$: $\Phi_i^{(j)} = \partial f / \partial x_i^{(j)}$, T is a positive parameter called temperature, and $\sigma_i(z; \beta) : \mathbb{R}^d \times \mathbb{R}_+ \to (0,1)$ gives the softmax function defined by

$$\sigma_i(z;\beta) = \frac{\exp(\beta z_i)}{\sum_{k=1}^d \exp(\beta z_k)} \tag{1.3}$$

with d being the dimension of input argument z. Note in passing that $y_i^{(j)}(t) \in (0,1)$ for arbitrary $t > 0$, i.e. $y(t) : [0, +\infty) \to (0,1)^n$. We are able to prove that the equilibrium points of the ODE system (1.2) represent the local optimum solutions of MCPP (1.1).