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A Weak Galerkin Method for the Coupled Darcy-Stokes Problem with a Nonstandard Transmission Condition on the Stokes Boundary

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Abstract. In this paper, we study a coupled problem of Darcy's law and Stokes equation with nonlinear slip boundary conditions. We derive a variational inequality for the coupled problem in detail. Then we introduce and analyze a weak Galerkin method to solve the coupled problem numerically. Under proper regularity assumptions, we obtain the optimal error estimate $\mathscr{O}(h)$ in newly defined h-norms. Finally, we give some numerical results to support the theoretical conclusions.

AMS subject classifications: 65N15, 65N30

Key words: Coupled Darcy-Stokes problem, variational inequality, weak Galerkin method, error estimate.

1. Introduction

The numerical simulation of the coupling of porous media flow and fluid flow has attracted the interest of many researchers due to its widespread applications in environment science, hydrology, biofluid dynamics and petroleum engineering [6, 33]. The simplest mathematical model is the classical coupled Darcy-Stokes problem, with the Darcy's law in the porous media region, the Stokes equations in the fluid region, and the standard transmission conditions on the boundaries [6, 11, 14]. While the classical coupled Darcy-Stokes model ignores the influence of boundary friction on the coupled system, for the Stokes flow, Fujita introduced slip boundary conditions of friction type to model blood flow in a vein of an arterial sclerosis patients, flow in avalanche of water and rocks, and flow in a canal with sherbet of mud and pebbles [10, 13]. A lot of numerical methods have been developed for the single Stokes flow with the slip conditions [10, 13, 22], but they ignore the interaction between the Stokes flow and its adjacent porous media flow. To better simulate the above phenomena, we introduce the following coupled Darcy-Stokes problem with the nonlinear slip conditions on the Stokes boundary.

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Let Ω_D and Ω_S be two bounded domains in $\mathbb{R}^2$ with $\Omega_D \cap \Omega_S = \emptyset$ and $\partial\Omega_D \cap \partial\Omega_S = \Gamma \neq \emptyset$. Define $\Gamma_D = \partial\Omega_D \setminus \overline{\Gamma}$, $\partial\Omega_S = \overline{\Gamma_{S_1}} \cup \overline{\Gamma_{S_2}} \cup \overline{\Gamma}$ with $\Gamma_{S_1} \cap \Gamma_{S_2} = \emptyset$ and $(\Gamma_{S_1} \cup \Gamma_{S_2}) \cap \Gamma = \emptyset$. Hereafter, we suppose the measures of Γ_{S_1} and Γ_D are nonzero. To better understand these domains and boundaries, we give a 2D geometric sketch in Fig. 1.

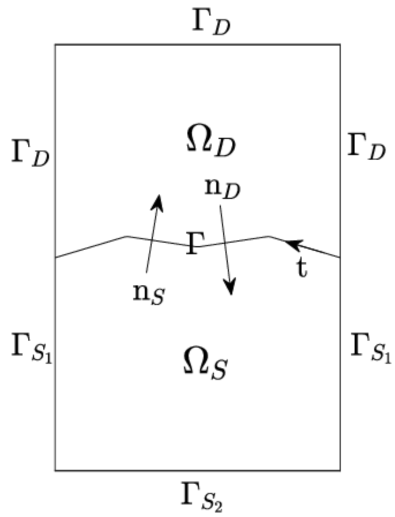

Figure 1: 2D geometric domains.

In Ω_D, we consider viscous fluid flows in a porous medium. According to the conservation of mass and the Darcy's law [11,14], we have

$$\nabla \cdot \boldsymbol{u}_D = f_D \quad \text{in } \Omega_D, \tag{1.1}$$

$$\mu K^{-1} \boldsymbol{u}_D + \nabla p_D = \mathbf{0} \quad \text{in } \Omega_D, \tag{1.2}$$

where $\boldsymbol{u}_D$ is the Darcy velocity field, p_D the Darcy pressure field, μ the Darcy viscosity, f_D is the external force density, and K^{-1} the inverse of the permeability tensor K with uniformly positive definite and uniformly bounded $K = K(\boldsymbol{x}) \in \mathbb{R}^{2\times 2}$, $\boldsymbol{x} \in \Omega_D$. Moreover, we consider the homogeneous Dirichlet boundary condition on Γ_D for the Darcy fluid

$$p_D = 0 \quad \text{on } \Gamma_D. \tag{1.3}$$

In Ω_S, we consider a viscous incompressible Stokes fluid flow with boundary friction. In view of the conservation of momentum and the conservation of mass, we have

$$-\nabla \cdot \big(2\nu e(\boldsymbol{u}_S)\big) + \nabla p_S = \boldsymbol{f}_S \quad \text{in } \Omega_S, \tag{1.4}$$

$$\nabla \cdot \boldsymbol{u}_S = 0 \quad \text{in } \Omega_S, \tag{1.5}$$

where $\boldsymbol{u}_S$ is the Stokes velocity field, p_S is the Stokes pressure field, ν is the Stokes viscosity,

$$e(\boldsymbol{u}_S) = \frac{1}{2}\big(\nabla \boldsymbol{u}_S + (\nabla \boldsymbol{u}_S)^\top\big)$$