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A Weak Galerkin Method for the Coupled Darcy-Stokes Problem with a Nonstandard Transmission Condition on the Stokes Boundary

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Abstract. In this paper, we study a coupled problem of Darcy's law and Stokes equation with nonlinear slip boundary conditions. We derive a variational inequality for the coupled problem in detail. Then we introduce and analyze a weak Galerkin method to solve the coupled problem numerically. Under proper regularity assumptions, we obtain the optimal error estimate $\mathscr{O}(h)$ in newly defined h-norms. Finally, we give some numerical results to support the theoretical conclusions.

AMS subject classifications: 65N15, 65N30

Key words: Coupled Darcy-Stokes problem, variational inequality, weak Galerkin method, error estimate.

1. Introduction

The numerical simulation of the coupling of porous media flow and fluid flow has attracted the interest of many researchers due to its widespread applications in environment science, hydrology, biofluid dynamics and petroleum engineering [6, 33]. The simplest mathematical model is the classical coupled Darcy-Stokes problem, with the Darcy's law in the porous media region, the Stokes equations in the fluid region, and the standard transmission conditions on the boundaries [6, 11, 14]. While the classical coupled Darcy-Stokes model ignores the influence of boundary friction on the coupled system, for the Stokes flow, Fujita introduced slip boundary conditions of friction type to model blood flow in a vein of an arterial sclerosis patients, flow in avalanche of water and rocks, and flow in a canal with sherbet of mud and pebbles [10, 13]. A lot of numerical methods have been developed for the single Stokes flow with the slip conditions [10, 13, 22], but they ignore the interaction between the Stokes flow and its adjacent porous media flow. To better simulate the above phenomena, we introduce the following coupled Darcy-Stokes problem with the nonlinear slip conditions on the Stokes boundary.

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Let Ω_D and Ω_S be two bounded domains in $\mathbb{R}^2$ with $\Omega_D \cap \Omega_S = \emptyset$ and $\partial\Omega_D \cap \partial\Omega_S = \Gamma \neq \emptyset$. Define $\Gamma_D = \partial\Omega_D \setminus \overline{\Gamma}$, $\partial\Omega_S = \overline{\Gamma_{S_1}} \cup \overline{\Gamma_{S_2}} \cup \overline{\Gamma}$ with $\Gamma_{S_1} \cap \Gamma_{S_2} = \emptyset$ and $(\Gamma_{S_1} \cup \Gamma_{S_2}) \cap \Gamma = \emptyset$. Hereafter, we suppose the measures of Γ_{S_1} and Γ_D are nonzero. To better understand these domains and boundaries, we give a 2D geometric sketch in Fig. 1.

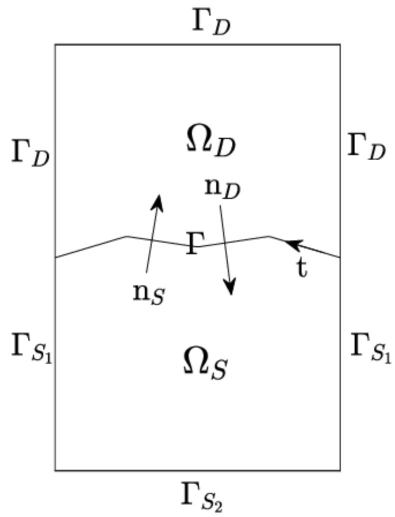

Figure 1: 2D geometric domains.

In Ω_D, we consider viscous fluid flows in a porous medium. According to the conservation of mass and the Darcy's law [11, 14], we have

$$\nabla \cdot \boldsymbol{u}_D = f_D \quad \text{in } \Omega_D, \tag{1.1}$$

$$\mu K^{-1} \boldsymbol{u}_D + \nabla p_D = \boldsymbol{0} \quad \text{in } \Omega_D, \tag{1.2}$$

where $\boldsymbol{u}_D$ is the Darcy velocity field, p_D the Darcy pressure field, μ the Darcy viscosity, f_D is the external force density, and K^{-1} the inverse of the permeability tensor K with uniformly positive definite and uniformly bounded $K = K(\boldsymbol{x}) \in \mathbb{R}^{2\times 2}$, $\boldsymbol{x} \in \Omega_D$. Moreover, we consider the homogeneous Dirichlet boundary condition on Γ_D for the Darcy fluid

$$p_D = 0 \quad \text{on } \Gamma_D. \tag{1.3}$$

In Ω_S, we consider a viscous incompressible Stokes fluid flow with boundary friction. In view of the conservation of momentum and the conservation of mass, we have

$$-\nabla \cdot \big(2\nu e(\boldsymbol{u}_S)\big) + \nabla p_S = \boldsymbol{f}_S \quad \text{in } \Omega_S, \tag{1.4}$$

$$\nabla \cdot \boldsymbol{u}_S = 0 \quad \text{in } \Omega_S, \tag{1.5}$$

where $\boldsymbol{u}_S$ is the Stokes velocity field, p_S is the Stokes pressure field, ν is the Stokes viscosity,

$$e(\boldsymbol{u}_S) = \frac{1}{2}\big(\nabla \boldsymbol{u}_S + (\nabla \boldsymbol{u}_S)^{\mathsf{T}}\big)$$

is the strain tensor and $\boldsymbol{f}_S$ is the external force density. Here, (1.5) represents the incompressibility condition. For simplicity, let

$$\boldsymbol{\sigma}(\boldsymbol{u}_S, p_S) = 2\nu e(\boldsymbol{u}_S) - p_S \mathbb{I}, \tag{1.6}$$

where $\mathbb{I}$ is the identity tensor in $\mathbb{R}^{2\times 2}$. Then, σ_t and σ_n are defined as

$$\sigma_t = \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S)\boldsymbol{n}_S \cdot \boldsymbol{t}, \quad \sigma_n = \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S)\boldsymbol{n}_S \cdot \boldsymbol{n}_S. \tag{1.7}$$

For the boundary friction conditions, we consider

$$\boldsymbol{u}_S = \boldsymbol{0} \qquad \text{on } \Gamma_{S_1}, \tag{1.8}$$

$$u_{S,n} = 0, \; |\sigma_t| \le g, \; \sigma_t u_{S,t} + g|u_{S,t}| = 0 \quad \text{on } \Gamma_{S_2}, \tag{1.9}$$

where

$$u_{S,n} = \boldsymbol{u}_S \cdot \boldsymbol{n}_S, \quad u_{S,t} = \boldsymbol{u}_S \cdot \boldsymbol{t}$$

with $\boldsymbol{n}_S$ and $\boldsymbol{t}$ being the external normal and tangential vectors along $\partial\Omega_S$ respectively, and $g \ge 0$ denotes the barrier or slip function on Γ_{S_2}. Moreover, (1.8) is the homogeneous Dirichlet boundary condition on Γ_{S_1}, $u_{S,n} = 0$ in (1.9) is the impermeability condition, and $|\sigma_t| \le g, \sigma_t u_{S,t} + g|u_{S,t}| = 0$ in (1.9) describes the friction — cf. [4, Eqs. (1.3)-(1.7)] for more details.

On the interface Γ, for the coupling of the Darcy flow and the Stokes flow, we consider the following transmission conditions:

$$(\boldsymbol{u}_S - \boldsymbol{u}_D) \cdot \boldsymbol{n}_S = 0 \quad \text{on } \Gamma, \tag{1.10}$$

$$\sigma_n + p_D = 0 \qquad \text{on } \Gamma, \tag{1.11}$$

$$\sigma_t + \kappa u_{S,t} = 0 \qquad \text{on } \Gamma, \tag{1.12}$$

where κ is a positive constant obtained from the experimental data. Moreover, (1.10) represents the conservation of mass, (1.11) represents the balance of normal forces, and (1.12) represents the Beavers-Joseph-Saffman law — cf. [14, Eqs. (1.4)-(1.6)] for more details.

Thus, we find the model (1.1)-(1.12) actually describes the coupling of the incompressible Stokes fluid with boundary friction and the Darcy fluid across an interface. Due to their wide applications in environment and science, lots of attention has been attracted to the numerical simulation for the coupled Darcy-Stokes problems. In [6], with the Raviart-Thomas elements, the estimate $\mathcal{O}(h^\delta)$, $\delta \in (0,1]$ was obtained for a coupled Darcy-Stokes problem with linear boundary conditions. In [33], with a staggered DG method, the optimal convergence estimates were obtained for the coupled nonlinear Darcy-Stokes problem with linear boundary conditions. As far as we know, for the problem (1.1)-(1.12), no convergence estimates are obtained at present.

Weak Galerkin (WG) methods are one type of highly flexible and robust finite element methods. Since first proposed by Wang and Ye [23], WG methods have been successfully applied to solve Stokes equations [25, 26, 29], Navier-Stokes equations [32], biharmonic equations [16,31], Maxwell equations [18], the quasistatic Maxwell viscoelastic model [27] and reaction-diffusion equations [12]. Besides these standard models, WG methods are also applied to some nonstandard models, such as elliptic equations on curved domains [15], elliptic equations with Newton boundary condition [20] and the wave equation with interface [3]. Moreover, some special techniques of WG methods, such as a posteriori error estimator for Stokes problem based on the auxiliary subspace techniques [30] and simple stabilizer free method with superconvergence [28] are also developed. These achievements and the advantages in flexibility and stability attract us to solve (1.1)-(1.12) with WG methods.

The rest of the paper is organized as follows. In Section 2, we consider the variational inequality of the coupled problem. In Section 3, we provide a WG method for the coupled problem. In Section 4, we give an error inequality for the error estimates of the WG method. In Section 5, we get the error estimates. Finally, we make some numerical experiments to support the conclusions.

2. Variational Inequality

2.1. Preliminaries and notations

For any $\mathscr{D} \in \mathbb{R}^2$ and $\mathscr{C} \in \mathbb{R}^1$, let $L^2(\mathscr{D})$ and $L^2(\mathscr{C})$ be the standard Lebesgue spaces — cf. [1, Chapter 2] for more details. Denote by $(\cdot,\cdot)_{\mathscr{D}}$ the inner product in $L^2(\mathscr{D})$, $[L^2(\mathscr{D})]^2$ and $[L^2(\mathscr{D})]^{2\times 2}$. Denote by $\langle\cdot,\cdot\rangle_{\mathscr{C}}$ the inner product in $L^2(\mathscr{C})$, $[L^2(\mathscr{C})]^2$ and $[L^2(\mathscr{C})]^{2\times 2}$. Then we have

$$
\begin{aligned}
&(u,v)_{\mathscr{D}} = \int_{\mathscr{D}} u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} && \text{for all} \quad u,v \in L^2(\mathscr{D}),\\
&\langle w,z\rangle_{\mathscr{C}} = \int_{\mathscr{C}} w(\boldsymbol{x})z(\boldsymbol{x})\mathrm{d}\boldsymbol{x} && \text{for all} \quad w,z \in L^2(\mathscr{C}),\\
&(\boldsymbol{u},\boldsymbol{v})_{\mathscr{D}} = \int_{\mathscr{D}} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} && \text{for all} \quad \boldsymbol{u},\boldsymbol{v} \in [L^2(\mathscr{D})]^2,\\
&\langle \boldsymbol{w},\boldsymbol{z}\rangle_{\mathscr{C}} = \int_{\mathscr{C}} \boldsymbol{w}(\boldsymbol{x})\cdot\boldsymbol{z}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} && \text{for all} \quad \boldsymbol{w},\boldsymbol{z} \in [L^2(\mathscr{C})]^2,\\
&(\mathbb{T}_1,\mathbb{T}_2)_{\mathscr{D}} = \int_{\mathscr{D}} \mathbb{T}_1(\boldsymbol{x}):\mathbb{T}_2(\boldsymbol{x})\mathrm{d}\boldsymbol{x} && \text{for all} \quad \mathbb{T}_1,\mathbb{T}_2 \in [L^2(\mathscr{D})]^{2\times 2},\\
&\langle\mathbb{T}_3,\mathbb{T}_4\rangle_{\mathscr{C}} = \int_{\mathscr{C}} \mathbb{T}_3(\boldsymbol{x}):\mathbb{T}_4(\boldsymbol{x})\mathrm{d}\boldsymbol{x} && \text{for all} \quad \mathbb{T}_3,\mathbb{T}_4 \in [L^2(\mathscr{C})]^{2\times 2}
\end{aligned} \tag{2.1}
$$

with " $\cdot$ " being the dot product of two vectors in $\mathbb{R}^2$ and " $:$ " being the sum of the products of the corresponding components of two matrices in $\mathbb{R}^{2\times 2}$. With the inner products $(\cdot,\cdot)_{\mathscr{D}}$ and $\langle\cdot,\cdot\rangle_{\mathscr{C}}$ above, we define the norms $\|\cdot\|_{\mathscr{D}}$ and $\|\cdot\|_{\mathscr{C}}$ as

$$
\|\cdot\|_{\mathscr{D}} = \sqrt{(\cdot,\cdot)_{\mathscr{D}}}, \quad \|\cdot\|_{\mathscr{C}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{C}}}. \tag{2.2}
$$

From (2.1), we have

$$(\mathbb{T}_1, \mathbb{T}_2)_{\mathscr{D}} = \left(\mathbb{T}_1^{\intercal}, \mathbb{T}_2^{\intercal}\right)_{\mathscr{D}}, \quad \left(\mathbb{T}_1 + \mathbb{T}_1^{\intercal}, \mathbb{T}_2\right)_{\mathscr{D}} = \frac{1}{2}\left(\mathbb{T}_1 + \mathbb{T}_1^{\intercal}, \mathbb{T}_2 + \mathbb{T}_2^{\intercal}\right)_{\mathscr{D}} \tag{2.3}$$

for all $\mathbb{T}_1, \mathbb{T}_2 \in [L^2(\mathscr{D})]^{2\times 2}$, where $\intercal$ denotes the transposition of a matrix or vector.

Given an integer $s \geq 0$, we use the standard Sobolev spaces $H^s(\mathscr{D})$ and $H^s(\mathscr{C})$ — cf. [1, Notation 1.2 and Chapter 3] for more details,

$$\begin{aligned}
H^s(\mathscr{D}) &= \left\{v \in L^2(\mathscr{D}) : \partial^\alpha v \in L^2(\mathscr{D}) \quad \text{for all} \quad |\alpha| \leq s\right\}, \\
H^s(\mathscr{C}) &= \left\{v \in L^2(\mathscr{C}) : \partial^\alpha v \in L^2(\mathscr{C}) \quad \text{for all} \quad |\alpha| \leq s\right\},
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ with α_1 and α_2 being nonnegative integers, and

$$\partial^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}}.$$

Denote by $(\cdot,\cdot)_{s,\mathscr{D}}$, $|\cdot|_{s,\mathscr{D}}$ and $\|\cdot\|_{s,\mathscr{D}}$ the inner-product, the seminorm and the norm in $H^s(\mathscr{D})$ respectively, and $\langle\cdot,\cdot\rangle_{s,\mathscr{C}}$, $|\cdot|_{s,\mathscr{C}}$, $\|\cdot\|_{s,\mathscr{C}}$ the inner-product, the seminorm, the norm in $H^s(\mathscr{C})$ respectively, with

$$\begin{aligned}
(u,v)_{s,\mathscr{D}} &= \sum_{0\leq|\alpha|\leq s} (\partial^\alpha u, \partial^\alpha v)_{\mathscr{D}}, & \langle w,z\rangle_{s,\mathscr{C}} &= \sum_{0\leq|\alpha|\leq s} \langle \partial^\alpha w, \partial^\alpha z\rangle_{\mathscr{C}}, \\
|v|^2_{s,\mathscr{D}} &= \sum_{|\alpha|=s} (\partial^\alpha v, \partial^\alpha v)_{\mathscr{D}}, & |w|^2_{s,\mathscr{C}} &= \sum_{|\alpha|=s} \langle \partial^\alpha w, \partial^\alpha w\rangle_{\mathscr{C}}, \\
\|v\|_{s,\mathscr{D}} &= \sum_{0\leq|\alpha|\leq s} |v|^2_{s,\mathscr{D}}, & \|w\|_{s,\mathscr{C}} &= \sum_{0\leq|\alpha|\leq s} |w|^2_{s,\mathscr{C}}
\end{aligned}$$

for any $u, v \in H^s(\mathscr{D})$ and $w, z \in H^s(\mathscr{C})$. In the subsequent content, we take

$$\mathscr{D} \in \{\Omega_S, \Omega_D, T\}, \quad \mathscr{C} \in \{\partial\Omega_S, \partial\Omega_D, \Gamma, \Gamma_{S_2}, \partial T, e\}$$

with T being an element in the partition and $e \subset \partial T$ being an edge of the element.

Next, we introduce a fundamental conclusion about elliptic variational inequalities.

Lemma 2.1 (cf. Glowinski [8, Theorem 4.1]). *Let V be a real Hilbert space. If $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bilinear, continuous and V-elliptic form on $V \times V$, $j(\cdot) : V \to \mathbb{R} \cup \{\infty\}$ is a convex lower semicontinuous and proper functional and $L(\cdot) : V \to \mathbb{R}$ is a continuous, linear functional, then the variational inequality*

$$a(u, v-u) + j(v) - j(u) \leq L(v-u) \quad \textit{for all} \quad v \in V$$

has a unique solution $u \in V$.

2.2. Variational inequality

This subsection introduces the derivation of the variational inequality for (1.1)-(1.12), and discusses the existence and uniqueness of the weak solution. According to the boundary conditions (1.3), (1.8) and $u_{S,n}=0$ in (1.9), we define

$$\begin{aligned} U_S^0 &= \left\{ \boldsymbol{v}_S \in [H^1(\Omega_S)]^2 : \boldsymbol{v}_S|_{\Gamma_{S_1}} = \boldsymbol{0},\ v_{S,n}|_{\Gamma_{S_2}} = 0 \right\}, \\ P_D^0 &= \left\{ q_D \in H^1(\Omega_D) : q_D|_{\Gamma_D} = 0 \right\}. \end{aligned}$$

Multiplying (1.1) by $q_D \in P_D^0$, integrating it over Ω_D, and applying the integration by parts yield

$$(f_D, q_D)_{\Omega_D} = (\nabla \cdot \boldsymbol{u}_D, q_D)_{\Omega_D} = -(\boldsymbol{u}_D, \nabla q_D)_{\Omega_D} + \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_D, q_D \rangle_{\partial\Omega_D}, \tag{2.4}$$

where $\boldsymbol{n}_D$ is the unit normal vector along $\partial\Omega_D$ with $\boldsymbol{n}_D = -\boldsymbol{n}_S$ on Γ.

Multiplying (1.4) by $\boldsymbol{v}_S \in U_S^0$, integrating it over Ω_S, and applying the integration by parts, (2.3) and (1.6) yields

$$\begin{aligned} (\boldsymbol{f_S}, \boldsymbol{v}_S)_{\Omega_S} &= -\big(\nabla \cdot (2\nu e(\boldsymbol{u}_S)), \boldsymbol{v}_S\big)_{\Omega_S} + (\nabla p_S, \boldsymbol{v}_S)_{\Omega_S} \\ &= 2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_S)\big)_{\Omega_S} - 2\nu\big\langle e(\boldsymbol{u}_S)\boldsymbol{n}_S, \boldsymbol{v}_S \big\rangle_{\partial\Omega_S} \\ &\quad - (\nabla \cdot \boldsymbol{v}_S, p_S)_{\Omega_S} + \langle p_S, \boldsymbol{v}_S \cdot \boldsymbol{n}_S \rangle_{\partial\Omega_S} \\ &= 2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_S)\big)_{\Omega_S} - (\nabla \cdot \boldsymbol{v}_S, p_S)_{\Omega_S} - \big\langle \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S)\boldsymbol{n}_S, \boldsymbol{v}_S \big\rangle_{\partial\Omega_S}. \end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5) to each other and applying $q_D \in P_D^0$ yields

$$\begin{aligned} &2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_S)\big)_{\Omega_S} - (p_S, \nabla \cdot \boldsymbol{v}_S)_{\Omega_S} \\ &\quad - \langle \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S)\boldsymbol{n}_S, \boldsymbol{v}_S \rangle_{\partial\Omega_S} \\ &\quad - (\boldsymbol{u}_D, \nabla q_D)_{\Omega_D} + \big\langle \boldsymbol{u}_D \cdot \boldsymbol{n}_D, q_D \big\rangle_\Gamma \\ &= \big(\boldsymbol{f}_S, \boldsymbol{v}_S\big)_{\Omega_S} + (f_D, q_D)_{\Omega_D}. \end{aligned} \tag{2.6}$$

From (1.2) and the transmission condition (1.10), we have

$$\begin{aligned} \boldsymbol{u}_D &= -\frac{1}{\mu} K \nabla p_D && \text{on } \Omega_D, \\ \boldsymbol{u}_D \cdot \boldsymbol{n}_D &= -\boldsymbol{u}_S \cdot \boldsymbol{n}_S && \text{on } \Gamma. \end{aligned} \tag{2.7}$$

From (1.7), the transmission conditions (1.11)-(1.12), (1.7) and $\boldsymbol{v}_S \in U_S^0$, we get

$$\begin{aligned} &\big\langle \boldsymbol{\sigma}(\boldsymbol{u}_S, p_S)\boldsymbol{n}_S, \boldsymbol{v}_S \big\rangle_{\partial\Omega_S} \\ &= \langle \sigma_n, \boldsymbol{v}_{S,n} \rangle_{\partial\Omega_S} + \langle \sigma_t, \boldsymbol{v}_{S,t} \rangle_{\partial\Omega_S} \\ &= \langle \sigma_n, \boldsymbol{v}_{S,n} \rangle_\Gamma + \langle \sigma_t, \boldsymbol{v}_{S,t} \rangle_\Gamma + \langle \sigma_t, \boldsymbol{v}_{S,t} \rangle_{\Gamma_{S_2}} \\ &= -\langle p_D, v_{S,n} \rangle_\Gamma - \kappa \langle u_{S,t}, v_{S,t} \rangle_\Gamma + \langle \sigma_t, \boldsymbol{v}_{S,t} \rangle_{\Gamma_{S_2}}. \end{aligned} \tag{2.8}$$

Combining (2.6)-(2.8) yields

$$\begin{aligned}
&2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_S)\big)_{\Omega_S} - (\nabla\cdot\boldsymbol{v}_S, p_S)_{\Omega_S} + \frac{1}{\mu}(K\nabla p_D, \nabla q_D)_{\Omega_D} \\
&\quad + \langle p_D, v_{S,n}\rangle_\Gamma - \langle u_{S,n}, q_D\rangle_\Gamma + \kappa\langle u_{S,t}, v_{S,t}\rangle_\Gamma \\
&= \big(\boldsymbol{f}_S, \boldsymbol{v}_S\big)_{\Omega_S} + (f_D, q_D)_{\Omega_D} + \langle \sigma_t, v_{S,t}\rangle_{\Gamma_{S_2}}.
\end{aligned}$$

Replacing $\boldsymbol{v}_S$ and q_D respectively by $\boldsymbol{v}_S - \boldsymbol{u}_S$ and $q_D - p_D$, and using the boundary condition (1.9), we get

$$\begin{aligned}
&2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_S - \boldsymbol{u}_S)\big)_{\Omega_S} - \big(\nabla\cdot(\boldsymbol{v}_S - \boldsymbol{u}_S), p_S\big)_{\Omega_S} + \frac{1}{\mu}\big(K\nabla p_D, \nabla(q_D - p_D)\big)_{\Omega_D} \\
&\quad + \langle p_D, v_{S,n} - u_{S,n}\rangle_\Gamma - \langle u_{S,n}, q_D - p_D\rangle_\Gamma + \kappa\langle u_{S,t}, v_{S,t} - u_{S,t}\rangle_\Gamma \\
&= (\boldsymbol{f}_S, \boldsymbol{v}_S - \boldsymbol{u}_S)_{\Omega_S} + (f_D, q_D - p_D)_{\Omega_D} + \langle \sigma_t, v_{S,t} - u_{S,t}\rangle_{\Gamma_{S_2}} \\
&\geq (\boldsymbol{f}_S, \boldsymbol{v}_S - \boldsymbol{u}_S)_{\Omega_S} + (f_D, q_D - p_D)_{\Omega_D} + \int_{\Gamma_{S_2}} g|u_{S,t}|\,\mathrm{d}s - \int_{\Gamma_{S_2}} g|v_{S,t}|\,\mathrm{d}s. \qquad (2.9)
\end{aligned}$$

Since $[H_0^1(\Omega_S)]^2 \subset U_S^0$, the estimate [7, (5.14)] gives

$$\sup_{\boldsymbol{v}_S\in U_S^0} \frac{(\nabla\cdot\boldsymbol{v}_S, q_S)_{\Omega_S}}{|\boldsymbol{v}_S|_{1,\Omega_S}} \geq \sup_{\boldsymbol{v}_S\in[H_0^1(\Omega_S)]^2} \frac{(\nabla\cdot\boldsymbol{v}_S, q_S)_{\Omega_S}}{|\boldsymbol{v}_S|_{1,\Omega_S}} \geq C\|q_S\|_{\Omega_S} \tag{2.10}$$

for all $q_S \in L_0^2(\Omega_S)$. From (2.10) and [7, Lemma 4.1], we find that $\nabla\cdot\boldsymbol{u}_S = 0$ in (1.5) is equivalent to

$$(\nabla\cdot\boldsymbol{u}_S, q_S)_{\Omega_S} = 0 \quad \text{for all} \quad q_S \in L_0^2(\Omega). \tag{2.11}$$

Then, in view of (2.9) and (2.11), we get the variational inequality for the coupled problem (1.1)-(1.12) reads as follows: Find $(\boldsymbol{u}_S, p_S, p_D) \in U_S^0 \times L_0^2(\Omega_S) \times P_D^0$ such that

$$\begin{aligned}
&a_S(\boldsymbol{u}_S, \boldsymbol{v}_S - \boldsymbol{u}_S) + b_S(\boldsymbol{v}_S - \boldsymbol{u}_S, p_S) + a_D(p_D, q_D - p_D) \\
&\quad + B(\boldsymbol{u}_S, \boldsymbol{v}_S - \boldsymbol{u}_S; p_D, q_D - p_D) + j(v_{S,t}) - j(u_{S,t}) \\
&\geq F(\boldsymbol{v}_S - \boldsymbol{u}_S, q_D - p_D) \qquad \text{for all} \quad (\boldsymbol{v}_S, q_D) \in U_S^0 \times P_D^0, \\
&b_S(\boldsymbol{u}_S, q_S) = 0 \qquad\qquad\qquad \text{for all} \quad q_S \in L_0^2(\Omega_S),
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
&L_0^2(\Omega_S) = \left\{ q \in L^2(\Omega_S) : \int_{\Omega_S} q\,\mathrm{d}x = 0 \right\}, \\
&a_S(\boldsymbol{u}_S, \boldsymbol{v}_S) = 2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_S)\big)_{\Omega_S}, \\
&a_D(p_D, q_D) = \frac{1}{\mu}(K\nabla p_D, \nabla q_D)_{\Omega_D}, \\
&b_S(\boldsymbol{v}_S, q_S) = -(\nabla\cdot\boldsymbol{v}_S, q_S)_{\Omega_S},
\end{aligned}$$

$$
\begin{aligned}
&B(\boldsymbol{u}_S,\boldsymbol{v}_S;p_D,q_D)=\langle p_D,\boldsymbol{v}_S\cdot\boldsymbol{n}_S\rangle_\Gamma-\langle \boldsymbol{u}_S\cdot\boldsymbol{n}_S,q_D\rangle_\Gamma+\kappa\langle u_{S,t},v_{S,t}\rangle_\Gamma,\\
&j(v_{S,t})=\int_{\Gamma_{S_2}} g|v_{S,t}|\,\mathrm{d}s,\\
&F(\boldsymbol{v}_S,q_S)=(\boldsymbol{f}_S,\boldsymbol{v}_S)_{\Omega_S}+(f_D,q_D)_{\Omega_D}.
\end{aligned}
$$

Next, we will simplify the variational inequality (2.12). Define

$$
X=\left\{\boldsymbol{v}_S\in U_S^0: b_S(\boldsymbol{v}_S,q_S)=0 \quad \text{for all} \quad q_S\in L_0^2(\Omega_S)\right\}
$$

and let $\mathscr{S}=X\times P_D^0$.

Lemma 2.2. *Define* $(\!(\cdot,\cdot)\!)_{\mathscr{S}}:\mathscr{S}\times\mathscr{S}\to\mathbb{R}$ *by*

$$
(\!(\zeta_1,\zeta_2)\!)_{\mathscr{S}}=\Big(e\big(\boldsymbol{v}_S^{(1)}\big),e\big(\boldsymbol{v}_S^{(2)}\big)\Big)_{\Omega_S}+\Big(\nabla q_D^{(1)},\nabla q_D^{(2)}\Big)_{\Omega_D}
$$

for all $\zeta_i=(\boldsymbol{v}_S^{(i)},q_D^{(i)})\in\mathscr{S}$, $i=1,2$. *Then* $(\!(\cdot,\cdot)\!)_{\mathscr{S}}$ *is an inner product on* $\mathscr{S}$, *and* $(\mathscr{S},(\!(\cdot,\cdot)\!)_{\mathscr{S}})$ *is a Hilbert space.*

Proof. For simplicity, we only prove the positive definiteness of $(\!(\cdot,\cdot)\!)_{\mathscr{S}}$. From (2.2), we have

$$
\|\cdot\|_{\Omega_S}^2=(\cdot,\cdot)_{\Omega_S},\quad \|\cdot\|_{\Omega_D}^2=(\cdot,\cdot)_{\Omega_D},
$$

where $(\cdot,\cdot)_{\Omega_i}$ $(i=S,D)$ are defined in (2.1). Suppose $(\!(\zeta,\zeta)\!)_{\mathscr{S}}=0$ for some $\zeta=(\boldsymbol{v}_S,q_D)\in\mathscr{S}$, then we arrive at

$$
\|e(\boldsymbol{v}_S)\|_{\Omega_S}^2+\|\nabla q_D\|_{\Omega_D}^2=0,
$$

which yields $e(\boldsymbol{v}_S)=\boldsymbol{0}$ and $\nabla q_D=0$. Due to [9, Lemma II.1], we have $\boldsymbol{v}_S=\boldsymbol{0}$ from $e(\boldsymbol{v}_S)=\boldsymbol{0}$. With the Poincaré-Friedrichs inequality — cf. [2, Ineq. (1.1)], we derive $q_D=0$ from $\nabla q_D=0$. Thus, $\zeta=(\boldsymbol{0},0)\in\mathscr{S}$ and $(\!(\cdot,\cdot)\!)_{\mathscr{S}}$ is an inner product.

Now, we prove that $(\mathscr{S},(\!(\cdot,\cdot)\!)_{\mathscr{S}})$ is a Hilbert space. Denote by $\|\cdot\|_{\mathscr{S}}$ the norm induced by the inner product $(\!(\cdot,\cdot)\!)_{\mathscr{S}}$. Suppose $\{\zeta_n\}_{n=1}^\infty$ $(\zeta_n=(\boldsymbol{v}_S^{(n)},q_D^{(n)}))$ is a Cauchy sequence in $\mathscr{S}$. Then for any $\epsilon>0$, there exists an integer $N(\epsilon)>0$ such that for all $m,n>N_\epsilon$,

$$
\|\zeta_m-\zeta_n\|_{\mathscr{S}}^2=\big\|e\big(\boldsymbol{v}_S^{(m)}\big)-e\big(\boldsymbol{v}_S^{(n)}\big)\big\|_{\Omega_S}^2+\big\|\nabla\big(q_D^{(m)}-q_D^{(n)}\big)\big\|_{\Omega_D}^2<\epsilon.
$$

According to [5, Theorem 3.1] and the Poincaré-Friedrichs inequality, there exist two constants $C_1>0$ and $C_2>0$ such that

$$
C_1\big\|\boldsymbol{v}_S^{(m)}-\boldsymbol{v}_S^{(n)}\big\|_{1,\Omega_S}\le\big\|e\big(\boldsymbol{v}_S^{(m)}\big)-e\big(\boldsymbol{v}_S^{(n)}\big)\big\|_{\Omega_S}<\sqrt{\epsilon/2}, \tag{2.13}
$$

$$
C_2\big\|q_D^{(m)}-q_D^{(n)}\big\|_{1,\Omega_D}\le\big\|\nabla\big(q_D^{(m)}-q_D^{(n)}\big)\big\|_{\Omega_D}<\sqrt{\epsilon/2}. \tag{2.14}
$$

Thus $\{\boldsymbol{v}_S^{(n)}\}_{n=1}^\infty$ and $\{q_D^{(n)}\}_{n=1}^\infty$ are Cauchy sequences in Banach spaces $(X,\|\cdot\|_{1,\Omega_S})$ and $(P_D^0,\|\cdot\|_{1,\Omega_D})$ respectively. Then there exists $(\boldsymbol{v}_S,q_D)\in\mathscr{S}$ and two positive integers N_1,N_2 such that (2.13) holds for all $n>N_1$ with $\boldsymbol{v}_S^{(m)}$ replaced with $\boldsymbol{v}_S$ and (2.14) holds for all $n>N_2$

with $q_D^{(m)}$replaced with q_D. Let $\zeta = (\boldsymbol{v}_S, q_D) \in \mathscr{S}$, then we can obtain $\|\zeta - \zeta_n\|_{\mathscr{S}}^2 < \epsilon$ for all $n > \max\{N_1, N_2\}$. Therefore, $(\mathscr{S}, \|\cdot\|_{\mathscr{S}})$ is a Banach space, and $(\mathscr{S}, ((\zeta_1, \zeta_2))_{\mathscr{S}})$ is a Hilbert space. $\square$

Lemma 2.3. *The problem* (2.12) *is equivalent to finding* $\varsigma = (\boldsymbol{u}_S, p_D) \in \mathscr{S}$ *such that*

$$\mathbb{B}(\varsigma, \zeta - \varsigma) + j(v_{S,t}) - j(u_{S,t}) \geq \mathbb{F}(\zeta - \varsigma) \quad \textit{for all} \quad \zeta = (\boldsymbol{v}_S, q_D) \in \mathscr{S}, \tag{2.15}$$

where $\mathbb{B}(\cdot,\cdot) : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ *and* $\mathbb{F} \in \mathscr{S}'$ *are defined by*

$$\begin{aligned}
&\mathbb{B}(\varsigma, \zeta) = a_S(\boldsymbol{u}_S, \boldsymbol{v}_S) + a_D(p_D, q_D) + B(\boldsymbol{u}_S, \boldsymbol{v}_S; p_D, q_D), \\
&\mathbb{F}(\zeta) = F(\boldsymbol{v}_S, q_D).
\end{aligned}$$

Proof. Following the inf-sup condition for $(\nabla \cdot \boldsymbol{v}_S, q_S)_{\Omega_S}$, cf. [7, Corollary 2.4 and Lemma 4.1], there exists a constant $C > 0$ such that

$$\inf_{q_S \in L_0^2(\Omega_S)} \sup_{\boldsymbol{v}_S \in H^1(\Omega_S)} \frac{(\nabla \cdot \boldsymbol{v}_S, q_S)_{\Omega_S}}{\|q_S\|_{\Omega_S} \|\boldsymbol{v}_S\|_{1,\Omega_S}} \geq C.$$

Thus (2.12) is equivalent to finding $(\boldsymbol{u}_S, p_D) \in X \times P_D^0$ satisfying

$$\begin{aligned}
&a_S(\boldsymbol{u}_S, \boldsymbol{v}_S - \boldsymbol{u}_S) + a_D(p_D, q_D - p_D) \\
&\quad + B(\boldsymbol{u}_S, \boldsymbol{v}_S - \boldsymbol{u}_S; p_D, q_D - p_D) + j(v_{S,t}) - j(u_{S,t}) \\
&\geq F(\boldsymbol{v}_S - \boldsymbol{u}_S, q_D - p_D) \quad \text{for all} \quad (\boldsymbol{v}_S, q_D) \in X \times P_D^0.
\end{aligned} \tag{2.16}$$

With the definition of $\mathbb{B}(\cdot,\cdot)$ and $\mathbb{F}(\cdot)$, (2.16) is equivalent to (2.15). Consequently, (2.12) is equivalent to (2.15). $\square$

Lemma 2.4. *For* $\mathbb{B}(\cdot,\cdot)$ *defined in Lemma* 2.3*, there exist two constants* $\alpha, \beta > 0$ *such that*

$$\mathbb{B}(\zeta, \zeta) \geq \alpha \|\zeta\|_{\mathscr{S}}^2, \tag{2.17}$$

$$|\mathbb{B}(\varsigma, \zeta)| \leq \beta \|\varsigma\|_{\mathscr{S}} \|\zeta\|_{\mathscr{S}} \tag{2.18}$$

for all $\varsigma, \zeta \in \mathscr{S}$.

Proof. From [1, Theorems 5.22, 5.36], there exist extension operators $E_i : H^1(\Omega_i) \to H^1(\mathbb{R}^2)$ $(i = S, D)$ such that

$$\|E_i v_i\|_{\partial\Omega_i} \leq C \|v_i\|_{1,\Omega_i} \quad \text{for all} \quad v_i \in H^1(\Omega_i). \tag{2.19}$$

Let

$$\boldsymbol{E}_S = (E_S, E_S)^{\intercal} : [H^1(\Omega_S)]^2 \to [H^1(\mathbb{R}^2)]^2.$$

For any $(\boldsymbol{v}_S, q_D) \in \mathscr{S} \subset [H^1(\Omega_S)]^2 \times H^1(\Omega_D)$, we extend the definition by

$$\boldsymbol{v}_S(\boldsymbol{x}) = \boldsymbol{E}_S \boldsymbol{v}_S(\boldsymbol{x}) \quad \text{a.e. on } \Gamma, \tag{2.20}$$

$$q_D(\boldsymbol{x}) = E_D q_D(\boldsymbol{x}) \quad \text{a.e. on } \Gamma. \tag{2.21}$$

Then from (2.19)-(2.21), we derive

$$\|\boldsymbol{v}_S\|_\Gamma = \|\boldsymbol{E}_S\boldsymbol{v}_S\|_\Gamma \le \|\boldsymbol{E}_S\boldsymbol{v}_S\|_{\partial\Omega_S} \le C\|\boldsymbol{v}_S\|_{1,\Omega_S}, \tag{2.22}$$

$$\|q_D\|_\Gamma = \|E_D q_D\|_\Gamma \le \|E_D q_D\|_{\partial\Omega_D} \le C\|q_D\|_{1,\Omega_D}. \tag{2.23}$$

From [5, (2.29) and Theorem 3.3 of Chapter 3], and the Poincaré-Friedrichs inequality, we have

$$\|\boldsymbol{v}_S\|_{1,\Omega_S} \le C\|e(\boldsymbol{v}_S)\|_{\Omega_S}, \quad \|q_D\|_{1,\Omega_D} \le C\|\nabla q_D\|_{\Omega_D} \tag{2.24}$$

for any $(\boldsymbol{v}_S, q_D) \in \mathscr{S}$. With (2.22)-(2.24), there exist constants $C_S, C_D > 0$ such that

$$\|\boldsymbol{v}_S\|_\Gamma \le C_S\,\|e(\boldsymbol{v}_S)\|_{\Omega_S}, \quad \|q_D\|_\Gamma \le C_D\|\nabla q_D\|_{\Omega_D} \tag{2.25}$$

for any $(\boldsymbol{v}_S, q_D) \in \mathscr{S}$.

Since $K(\boldsymbol{x}) \in \mathbb{R}^{2\times 2}$ with $\boldsymbol{x} \in \Omega_D$ is uniformly positive definite and uniformly bounded, there exist two constants M_K, m_K with $M_K \ge m_K > 0$ such that

$$\begin{aligned} &\boldsymbol{v}(\boldsymbol{x})^\intercal K(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}) \ge m_K \boldsymbol{v}(\boldsymbol{x})^\intercal \boldsymbol{v}(\boldsymbol{x}), \\ &|K(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})| \le M_K\sqrt{\boldsymbol{v}(\boldsymbol{x})^\intercal\boldsymbol{v}(\boldsymbol{x})} \end{aligned} \tag{2.26}$$

for any $\boldsymbol{v}(\boldsymbol{x}) \in \mathbb{R}^2$ with $\boldsymbol{x} \in \Omega_D$.

Let $\varsigma = (\boldsymbol{u}_S, p_D) \in \mathscr{S}$ and $\zeta = (\boldsymbol{v}_S, q_D) \in \mathscr{S}$. From (2.25) and (2.26), we have

$$\begin{aligned} \mathbb{B}(\zeta,\zeta) &= 2\nu\|e(\boldsymbol{v}_S)\|^2_{\Omega_S} + \frac{1}{\mu}(K\nabla q_D, \nabla q_D)_{\Omega_D} + \kappa\langle \boldsymbol{v}_S\cdot\boldsymbol{t}, \boldsymbol{v}_S\cdot\boldsymbol{t}\rangle_\Gamma \ge \alpha\,\|\zeta\|^2_{\mathscr{S}}, \\ |\mathbb{B}(\varsigma,\zeta)| &\le \left(2\nu\|e(\boldsymbol{u}_S)\|^2_{\Omega_S} + \frac{1}{\mu}M_K\,\|\nabla p_D\|^2_{\Omega_D} + \|p_D\|^2_\Gamma + (\kappa+1)\,\|\boldsymbol{u}_S\|^2_\Gamma\right)^{1/2} \\ &\quad\times\left(2\nu\|e(\boldsymbol{v}_S)\|^2_{\Omega_S} + \frac{1}{\mu}M_K\,\|\nabla q_D\|^2_{\Omega_D} + (\kappa+1)\,\|\boldsymbol{v}_S\|^2_\Gamma + \|q_D\|^2_\Gamma\right)^{1/2} \\ &\le \beta\,\|\varsigma\|_{\mathscr{S}}\,\|\zeta\|_{\mathscr{S}}, \end{aligned}$$

where

$$\alpha = \min\left\{2\nu, \frac{1}{\mu}m_K\right\}, \quad \beta = \max\left\{2\nu + (\kappa+1)C_S^2, \frac{1}{\mu}M_K + C_D^2\right\}.$$

Thus (2.17) and (2.18) get proved. □

Theorem 2.1. *If $g \in L^2(\Gamma_{S_2})$, the problem* (2.12) *has a unique solution $(\boldsymbol{u}_S, p_S, p_D) \in U_S^0 \times L_0^2(\Omega_S) \times P_D^0$.*

Proof. In view of Lemma 2.3, we only need to prove that (2.15) has a unique solution $\varsigma = (\boldsymbol{u}_S, p_D) \in \mathscr{S}$. It follows from (2.25) that

$$|j(v_{S,t}) - j(u_{S,t})| \le \int_{\Gamma_{S_2}} g|v_{S,t} - u_{S,t}|\,\mathrm{d}s \le C_S\,\|g\|_{\Gamma_{S_2}}\,\|e(\boldsymbol{v}_S) - e(\boldsymbol{u}_S)\|_{\Omega_S},$$

thus $j(\cdot)$ is continuous on $\mathscr{S}$. And $j(\cdot)$ is convex and proper on $\mathscr{S}$ with the properties of absolute value. From Lemma 2.1, (2.15) has a unique solution $\varsigma = (\boldsymbol{u}_S, p_D) \in \mathscr{S}$. □

3. WG Method

Let $\mathcal{T}_{h_i}$ $(i=S,D)$ be the regular polygonal partitions of Ω_i that is aligned with Γ, and ε_{h_i} be the set of all edges in $\mathcal{T}_{h_i}$ and $\varepsilon_{h_i}^I=\varepsilon_{h_i}\setminus\partial\Omega_i$. For any $T\in\mathcal{T}_{h_i}$, denote by h_T the diameter of T. Then let $h_i=\max_{T\in\mathcal{T}_{h_i}}h_T$ and $h=\max_{i\in\{S,D\}}h_i$. Define

$$
\begin{aligned}
&\mathcal{W}_{h_i}=\left\{\{v_0,v_b\}:v_0|_T\in L^2(T)\quad\text{for all}\quad T\in\mathcal{T}_{h_i};v_b|_e\in L^2(e)\quad\text{and all}\quad e\in\varepsilon_{h_i}\right\},\\
&[\mathcal{W}_{h_i}]^2=\left\{\{\boldsymbol{v}_0,\boldsymbol{v}_b\}:\boldsymbol{v}_0|_T\in[L^2(T)]^2\quad\text{for all}\quad T\in\mathcal{T}_{h_i};\boldsymbol{v}_b|_e\in[L^2(e)]^2\quad\text{and all}\quad e\in\varepsilon_{h_i}\right\},\\
&\boldsymbol{M}_{h_S}=\left\{\boldsymbol{\omega}_S:\boldsymbol{\omega}_S|_T\in[P_0(T)]^{2\times2}\quad\text{for all}\quad T\in\mathcal{T}_{h_S}\right\},\\
&\overline{\boldsymbol{M}}_{h_S}=\left\{\boldsymbol{\omega}_S\in\boldsymbol{M}_{h_S}:\boldsymbol{\omega}_S=\boldsymbol{\omega}_S^{\intercal}\right\},\\
&\boldsymbol{U}_{h_S}=\left\{\{\boldsymbol{v}_0,\boldsymbol{v}_b\}:\boldsymbol{v}_0|_T\in[P_1(T)]^2\quad\text{for all}\quad T\in\mathcal{T}_{h_S};\boldsymbol{v}_b|_e\in[P_1(e)]^2\quad\text{and all}\quad e\in\varepsilon_{h_S}\right\},\\
&\boldsymbol{U}_{h_S}^0=\left\{\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in\boldsymbol{U}_{h_S}:\boldsymbol{v}_b|_{\Gamma_{S_1}}=\boldsymbol{0};\boldsymbol{v}_b\cdot\boldsymbol{n}_S|_{\Gamma_{S_2}}=0\right\},\\
&\boldsymbol{U}_{h_D}=\left\{\boldsymbol{v}_D:\boldsymbol{v}_D|_T\in[P_0(T)]^2\quad\text{for all}\quad T\in\mathcal{T}_{h_D}\right\},\\
&P_{h_D}=\left\{\{q_0,q_b\}:q_0|_T\in P_1(T)\quad\text{for all}\quad T\in\mathcal{T}_{h_D};q_b|_e\in P_0(e)\quad\text{and all}\quad e\in\varepsilon_{h_D}\right\},\\
&P_{h_D}^0=\left\{\{q_0,q_b\}\in P_{h_D}:q_b|_{\Gamma_D}=0\right\},\\
&P_{h_S}=\left\{q_S:q_S|_T\in P_0(T)\quad\text{for all}\quad T\in\mathcal{T}_{h_S}\right\},\\
&P_{h_S}^0=\left\{q_S\in P_{h_S}:q_S\in L_0^2(\Omega_S)\right\}.
\end{aligned}
$$

Definition 3.1. *Given any $q_D=\{q_0,q_b\}\in\mathcal{W}_{h_D}$, define $\nabla_w q_D\in\boldsymbol{U}_{h_D}$ such that*

$$(\nabla_w q_D,\boldsymbol{v}_D)_T=-(q_0,\nabla\cdot\boldsymbol{v}_D)_T+\langle q_b,\boldsymbol{v}_D\cdot\mathbf{n}\rangle_{\partial T}\tag{3.1}$$

for all $T\in\mathcal{T}_{h_D}$ and $\boldsymbol{v}_D\in[P_0(T)]^2$.

Definition 3.2. *Given any $\boldsymbol{v}_S=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in[\mathcal{W}_{h_S}]^2$, define $\nabla_w\cdot\boldsymbol{v}_S\in P_{h_S}$ such that*

$$(\nabla_w\cdot\boldsymbol{v}_S,q_S)_T=-(\boldsymbol{v}_0,\nabla q_S)_T+\langle\boldsymbol{v}_b\cdot\mathbf{n},q_S\rangle_{\partial T}\tag{3.2}$$

for any $T\in\mathcal{T}_{h_S}$ and $q_S\in P_0(T)$.

Definition 3.3. *Given any $\boldsymbol{v}_S=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in[\mathcal{W}_{h_S}]^2$, define $\nabla_w\boldsymbol{v}_S\in\boldsymbol{M}_{h_S}$ such that*

$$(\nabla_w\boldsymbol{v}_S,\boldsymbol{\omega}_S)_T=-(\boldsymbol{v}_0,\nabla\cdot\boldsymbol{\omega}_S)_T+\langle\boldsymbol{v}_b,\boldsymbol{\omega}_S\boldsymbol{n}\rangle_{\partial T}\tag{3.3}$$

for all $T\in\mathcal{T}_{h_S}$ and $\boldsymbol{\omega}_S\in[P_0(T)]^{2\times2}$.

Remark 3.1. Let

$$e_w(\boldsymbol{v}_S)=\frac{1}{2}\big(\nabla_w\boldsymbol{v}_S+(\nabla_w\boldsymbol{v}_S)^{\intercal}\big)$$

for any $\boldsymbol{v}_S=\{\boldsymbol{v}_0,\boldsymbol{v}_b\}\in[\mathcal{W}_{h_S}]^2$. From Definition 3.3, for all $T\in\mathcal{T}_{h_S}$ and $\boldsymbol{\Xi}\in[P_0(T)]^{2\times2}$, we have

$$\big(e_w(\boldsymbol{v}_S),\boldsymbol{\Xi}\big)_T=-\big(\boldsymbol{v}_0,\nabla\cdot\overline{\boldsymbol{\Xi}}\big)_T+\big\langle\boldsymbol{v}_b,\overline{\boldsymbol{\Xi}}\boldsymbol{n}\big\rangle_{\partial T},$$

where $\overline{\Xi} = (\Xi + \Xi^{\mathsf{T}})/2$. Thus $e_w(\boldsymbol{v}_S)$ is uniquely determined by the following equation:

$$\big(e_w(\boldsymbol{v}_S), \Xi\big)_T = -(\boldsymbol{v}_0, \nabla \cdot \Xi)_T + \langle \boldsymbol{v}_b, \Xi \boldsymbol{n}\rangle_{\partial T}$$

for all $T \in \mathscr{T}_{h_S}$ and $\Xi \in \overline{\boldsymbol{M}}_{h_S}$.

Next, we introduce local L^2-projection operators. For any function $\chi \in L^2(\Omega_i) \cup \mathbf{L}^2(\Omega_i) \cup \mathbb{L}^2(\Omega_i)$ $(i = S, D)$, with $\mathbf{L}^2(\Omega_i) = [L^2(\Omega_i)]^2$ and $\mathbb{L}^2(\Omega_i) = [L^2(\Omega_i)]^{2\times 2}$, we define $\boldsymbol{Q}_0, \boldsymbol{Q}_b, Q_0$, Q_b and $\mathbb{Q}_h$ as

$$\begin{aligned}
&(\boldsymbol{Q}_0\chi - \chi, \boldsymbol{\phi})_T = 0 && \text{for all} && \boldsymbol{\phi} \in [P_1(T)]^2 && \text{and all} && T \in \mathscr{T}_{h_S},\\
&\langle \boldsymbol{Q}_b\chi - \chi, \boldsymbol{\psi}\rangle_e = 0 && \text{for all} && \boldsymbol{\psi} \in [P_1(e)]^2 && \text{and all} && e \in \varepsilon_{h_S},\\
&(Q_0\chi - \chi, \phi)_T = 0 && \text{for all} && \phi \in P_1(T) && \text{and all} && T \in \mathscr{T}_{h_D},\\
&\langle Q_b\chi - \chi, \psi\rangle_e = 0 && \text{for all} && \psi \in P_0(e) && \text{and all} && e \in \varepsilon_{h_D},\\
&(\mathbb{Q}_h\chi - \chi, \theta)_T = 0 && \text{for all} && \theta \in P_0(T) && \text{and all} && T \in \mathscr{T}_{h_S} \cup \mathscr{T}_{h_S},
\end{aligned} \tag{3.4}$$

and define $\boldsymbol{Q}_h = \{\boldsymbol{Q}_0, \boldsymbol{Q}_b\}$, $Q_h = \{Q_0, Q_b\}$.

Lemma 3.1 (cf. Mu *et al.* [17, Lemma 3.5])**.** *For $\boldsymbol{Q}_h, Q_h$ and $\mathbb{Q}_h$ defined in* (3.4)*, we have*

$$\begin{aligned}
&\nabla_w(Q_h\phi) = \mathbb{Q}_h(\nabla\phi) && \textit{for all} && \phi \in H^1(T) && \textit{and all} && T \in \mathscr{T}_{h_D},\\
&e_w(\boldsymbol{Q}_h\boldsymbol{\phi}) = \mathbb{Q}_h e(\boldsymbol{\phi}) && \textit{for all} && \boldsymbol{\phi} \in [H^1(T)]^2 && \textit{and all} && T \in \mathscr{T}_{h_S},\\
&\nabla_w \cdot (\boldsymbol{Q}_h\boldsymbol{\phi}) = \mathbb{Q}_h(\nabla \cdot \boldsymbol{\phi}) && \textit{for all} && \boldsymbol{\phi} \in H_{\rm d}(T) && \textit{and all} && T \in \mathscr{T}_{h_S},
\end{aligned}$$

where

$$H_{\rm d}(T) = \left\{\boldsymbol{w} \in [L^2(T)]^2 : \nabla \cdot \boldsymbol{w} \in L^2(T)\right\}.$$

For any $\boldsymbol{u}_S = \{\boldsymbol{u}_0, \boldsymbol{u}_b\} \in [\mathscr{W}_{h_S}]^2$, $\boldsymbol{v}_S = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in [\mathscr{W}_{h_S}]^2$, $q_S \in P_h$, $p_D = \{p_0, p_b\} \in \mathscr{W}_{h_D}$ and $q_D = \{q_0, q_b\} \in \mathscr{W}_{h_D}$, define the following multilinear forms:

$$\begin{aligned}
&s_S(\boldsymbol{u}_S, \boldsymbol{v}_S) = \sum_{T\in\mathscr{T}_{h_S}} h_T^{-1}\langle \boldsymbol{u}_0 - \boldsymbol{u}_b, \boldsymbol{v}_0 - \boldsymbol{v}_b\rangle_{\partial T},\\
&a_S^h(\boldsymbol{u}_S, \boldsymbol{v}_S) = 2\nu \sum_{T\in\mathscr{T}_{h_S}} \big(e_w(\boldsymbol{u}_S), e_w(\boldsymbol{v}_S)\big)_T + 2\nu s_S(\boldsymbol{u}_S, \boldsymbol{v}_S),\\
&b_S^h(\boldsymbol{v}_S, q_S) = -\sum_{T\in\mathscr{T}_{h_S}} (\nabla_w \cdot \boldsymbol{v}_S, q_S)_T\,,\\
&s_D(p_D, q_D) = \sum_{T\in\mathscr{T}_{h_D}} h_T^{-1}\langle Q_b p_0 - p_b, Q_b q_0 - q_b\rangle_{\partial T},\\
&a_D^h(p_D, q_D) = \frac{1}{\mu}\sum_{T\in\mathscr{T}_{h_D}} (K\nabla_w p_D, \nabla_w q_D)_T + \frac{1}{\mu} m_K s_D(p_D, q_D),\\
&B^h(\boldsymbol{u}_S, \boldsymbol{v}_S; p_D, q_D) = \langle p_b, \boldsymbol{v}_b \cdot \boldsymbol{n}_S\rangle_\Gamma + \kappa\langle u_{b,t}, v_{b,t}\rangle_\Gamma - \langle \boldsymbol{u}_b \cdot \boldsymbol{n}_S, q_b\rangle_\Gamma,\\
&F^h(\boldsymbol{v}_S, q_D) = (\boldsymbol{f}_S, \boldsymbol{v}_0)_{\Omega_S} + (f_D, q_0)_{\Omega_D},\\
&j^h(v_{S,t}) = \int_{\Gamma_{S_2}} g|v_{b,t}|\,{\rm d}s,
\end{aligned} \tag{3.5}$$

where $m_K > 0$ is given in (2.26). Then, the WG scheme for the coupled problem (1.1)-(1.12) reads as follows: Find $(\boldsymbol{u}_S^h, p_S^h, p_D^h) \in \boldsymbol{U}_{h_S}^0 \times P_{h_S}^0 \times P_{h_D}^0$ such that

$$\begin{aligned} &a_S^h\big(\boldsymbol{u}_S^h, \boldsymbol{v}_S - \boldsymbol{u}_S^h\big) + b_S^h\big(\boldsymbol{v}_S - \boldsymbol{u}_S^h, p_S^h\big) + a_D^h\big(p_D^h, q_D - p_D^h\big) \\ &\quad + B^h\big(\boldsymbol{u}_S^h, \boldsymbol{v}_S - \boldsymbol{u}_S^h; p_D^h, q_D - p_D^h\big) + j^h(v_{S,t}) - j^h\big(u_{S,t}^h\big) \\ &\geq F^h\big(\boldsymbol{v}_S - \boldsymbol{u}_S^h, q_D - p_D^h\big) \quad \text{for all} \quad (\boldsymbol{v}_S, q_D) \in \boldsymbol{U}_{h_S}^0 \times P_{h_D}^0, \end{aligned} \tag{3.6}$$

$$b_S^h\big(\boldsymbol{u}_S^h, q_S\big) = 0 \qquad \text{for all} \quad q_S \in P_{h_S}^0. \tag{3.7}$$

Lemma 3.2. *Define* $\|\cdot\|_{h_S}$ *and* $\|\cdot\|_{h_D}$ *by*

$$\|\boldsymbol{v}_S\|_{h_S}^2 = \sum_{T \in \mathscr{T}_{h_S}} \|e_w(\boldsymbol{v}_S)\|_T^2 + s_S(\boldsymbol{v}_S, \boldsymbol{v}_S) \qquad \textit{for all} \quad \boldsymbol{v}_S \in \boldsymbol{U}_{h_S},$$

$$\|q_D\|_{h_D}^2 = \sum_{T \in \mathscr{T}_{h_D}} \|\nabla_w q_D\|_T^2 + s_D(q_D, q_D) \qquad \textit{for all} \quad q_D \in P_{h_D}^0,$$

where $s_S(\cdot,\cdot)$ *and* $s_D(\cdot,\cdot)$ *are defined in* (3.5). *Then* $\|\cdot\|_{h_S}$ *is a norm in* $\boldsymbol{U}_{h_S}^0$, *and* $\|\cdot\|_{h_D}$ *is a norm in* $P_{h_D}^0$.

Proof. From [25, Section 4], we know $\|\cdot\|_{h_D}$ is a norm in $P_{h_D}^0$. Thus, we only need to prove $\|\cdot\|_{h_S}$ is a norm in $\boldsymbol{U}_{h_S}^0$. For simplicity, we only prove the positive definiteness of $\|\cdot\|_{h_S}$ in $\boldsymbol{U}_{h_S}^0$. Assume that $\|\boldsymbol{v}_S\|_{h_S} = 0$ with $\boldsymbol{v}_S = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in \boldsymbol{U}_{h_S}^0$. Then we have

$$e_w(\boldsymbol{v}_S)|_T = \boldsymbol{0}, \quad \boldsymbol{v}_0|_e = \boldsymbol{v}_b|_e$$

for all $T \in \mathscr{T}_{h_S}$ and $e \subset \partial T$. In view of Remark 3.1, we derive

$$\begin{aligned} 0 = \big(e_w(\boldsymbol{v}_S), \boldsymbol{\omega}\big)_T &= -(\boldsymbol{v}_0, \nabla \cdot \boldsymbol{\omega})_T + \langle \boldsymbol{v}_b, \boldsymbol{\omega}\boldsymbol{n}_S\rangle_{\partial T} \\ &= (\nabla \boldsymbol{v}_0, \boldsymbol{\omega})_T + \langle \boldsymbol{v}_b - \boldsymbol{v}_0, \boldsymbol{\omega}\boldsymbol{n}_S\rangle_{\partial T} = \big(e(\boldsymbol{v}_0), \boldsymbol{\omega}\big)_T \end{aligned}$$

for all $T \in \mathscr{T}_{h_S}$ and $\omega \in \overline{\boldsymbol{M}}_{h_S}$. Taking $\boldsymbol{\omega} = e(\boldsymbol{v}_0)$, we get $e(\boldsymbol{v}_0) = \boldsymbol{0}$ for all $T \in \mathscr{T}_{h_S}$. Since $\boldsymbol{v}_0|_e = \boldsymbol{v}_b|_e$ for any $e \in \varepsilon_{h_S}$, we can obtain $\boldsymbol{v}_0 \in H^1(\Omega_S)^2$ and $\boldsymbol{v}_0|_{\Gamma_{S_1}} = \boldsymbol{0}$. Then due to Lemma 2.2, we find $\boldsymbol{v}_0 = \boldsymbol{0}$ on Ω_S. Thus, $\boldsymbol{v}_b|_e = \boldsymbol{0}$ for all $e \in \varepsilon_{h_S}$. □

Remark 3.2. Let

$$X_h = \Big\{\boldsymbol{v}_S \in \boldsymbol{U}_{h_S}^0 : b_S^h(\boldsymbol{v}_S, q_S) = 0 \quad \text{for all} \quad q_S \in P_{h_S}^0\Big\}$$

be a subspace of $\boldsymbol{U}_{h_S}^0$ and $\mathscr{S}_h = X_h \times P_{h_D}^0$. Define $(\!(\cdot,\cdot)\!)_{\mathscr{S}_h}$ by

$$(\!(\varsigma, \zeta)\!)_{\mathscr{S}_h} = \big(e_w(\boldsymbol{u}_S), e_w(\boldsymbol{v}_S)\big)_{\Omega_S} + s_S(\boldsymbol{u}_S, \boldsymbol{v}_S) + (\nabla_w p_D, \nabla_w q_D)_{\Omega_D} + s_D(p_D, q_D)$$

for all $\varsigma = (\boldsymbol{u}_S, p_D), \zeta = (\boldsymbol{v}_S, q_D) \in \mathscr{S}_h$. Then $(\!(\cdot,\cdot)\!)_{\mathscr{S}_h}$ is an inner product on $\mathscr{S}_h$ from Lemma 3.2. Let $\|\cdot\|_{\mathscr{S}_h}$ be the norm induced by the inner product $(\!(\cdot,\cdot)\!)_{\mathscr{S}_h}$. Since any finite dimensional normed space is a Banach space, $(\mathscr{S}_h, \|\cdot\|_{\mathscr{S}_h})$ is a Banach space, thus $(\mathscr{S}_h, (\!(\cdot,\cdot)\!)_{\mathscr{S}_h})$ is a Hilbert space.

Hereafter, we denote by C, C_1, C_2 the positive constants independent of h, which may differ in different cases.

Lemma 3.3 (cf. Wang & Ye [24, Lemma A.3])**.** *For any $T \in \mathscr{T}_{h_i}$ $(i = S, D)$ and $e \subset \partial T$, we have*

$$\|v\|_e^2 \leq C\left(h_T^{-1}\|v\|_T^2 + h_T\|\nabla v\|_T^2\right)$$

for all $v \in H^1(T)$.

Lemma 3.4. *For $\nabla_w \cdot$ defined in* (3.2)*, we have*

$$\inf_{q_S \in P_{h_S}^0} \sup_{\boldsymbol{v}_S \in \widetilde{\boldsymbol{U}}_{h_S}^0} \frac{(\nabla_w \cdot \boldsymbol{v}_S, q_S)_{\Omega_S}}{\|q_S\|_{\Omega_S} \|\boldsymbol{v}_S\|_{h_S}} \geq C.$$

Proof. Following [25, Lemma 4.3], for any $q_S \in P_{h_S}^0 \subset L_0^2(\Omega_S)$, there exists $\boldsymbol{v} \in H_0^1(\Omega_S)^2$ such that

$$\frac{(\nabla \cdot \boldsymbol{v}, q_S)_{\Omega_S}}{\|\boldsymbol{v}\|_{1,\Omega_S}} \geq C_1 \|q_S\|_{\Omega_S}. \tag{3.8}$$

Let $\boldsymbol{v}^* = \boldsymbol{Q}_h \boldsymbol{v} \in \widetilde{\boldsymbol{U}}_{h_S}^0$. According to Lemma 3.1, we find

$$\begin{aligned}
\sum_{T \in \mathscr{T}_{h_S}} \|e_w(\boldsymbol{v}^*)\|_T^2 &= \sum_{T \in \mathscr{T}_{h_S}} \|e_w(\boldsymbol{Q}_h \boldsymbol{v})\|_T^2 = \sum_{T \in \mathscr{T}_{h_S}} \|\mathbb{Q}_h e(\boldsymbol{v})\|_T^2 \\
&\leq \sum_{T \in \mathscr{T}_{h_S}} \|e(\boldsymbol{v})\|_T^2 \leq \|\boldsymbol{v}\|_{1,\Omega_S}^2,
\end{aligned} \tag{3.9}$$

$$(\nabla_w \cdot \boldsymbol{v}^*, q_S)_{\Omega_S} = \left(\mathbb{Q}_h(\nabla \cdot \boldsymbol{v}), q_S\right)_{\Omega_S} = (\nabla \cdot \boldsymbol{v}, q_S)_{\Omega_S}. \tag{3.10}$$

From [16, Lemma 4.1], we have

$$\sum_{T \in \mathscr{T}_{h_S}} \|\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v}\|_T^2 \leq Ch^2 \|\boldsymbol{v}\|_{1,\Omega_S}^2, \tag{3.11}$$

$$\sum_{T \in \mathscr{T}_{h_S}} \|\nabla(\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v})\|_T^2 \leq C\|\boldsymbol{v}\|_{1,\Omega_S}^2. \tag{3.12}$$

Since $Ch \leq h_T \leq h$ for any $T \in \mathscr{T}_{h_S}$ due to the regularity of the partition, and with (3.11), we get

$$\sum_{T \in \mathscr{T}_{h_S}} h_T^{-2} \|\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v}\|_T^2 \leq C\|\boldsymbol{v}\|_{1,\Omega_S}^2. \tag{3.13}$$

Due to (3.4), Lemma 3.3 and (3.12)-(3.13), we arrive at

$$\begin{aligned}
s_S(\boldsymbol{v}^*, \boldsymbol{v}^*) &= \sum_{T \in \mathscr{T}_{h_S}} h_T^{-1} \|\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{Q}_b \boldsymbol{v}\|_{\partial T}^2 \leq \sum_{T \in \mathscr{T}_{h_S}} h_T^{-1} \|\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v}\|_{\partial T}^2 \\
&\leq \sum_{T \in \mathscr{T}_{h_S}} C\left(h_T^{-2} \|\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v}\|_T^2 + \|\nabla(\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v})\|_T^2\right) \\
&\leq C \sum_{T \in \mathscr{T}_{h_S}} h_T^{-2} \|\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v}\|_T^2 + C \sum_{T \in \mathscr{T}_{h_S}} \|\nabla(\boldsymbol{Q}_0 \boldsymbol{v} - \boldsymbol{v})\|_T^2 \leq C_2 \|\boldsymbol{v}\|_{1,\Omega_S}^2.
\end{aligned} \tag{3.14}$$

Combining (3.9) and (3.14), we get

$$\|\boldsymbol{v}^*\|_{h_S} \leq \max\{1, C_2\} \|\boldsymbol{v}\|_{1,\Omega_S}^2. \tag{3.15}$$

With (3.8), (3.10) and (3.15), we can obtain

$$\sup_{\boldsymbol{v}_S \in \tilde{\boldsymbol{U}}_{h_S}^0} \frac{(\nabla_w \cdot \boldsymbol{v}_S, q_S)_{\Omega_S}}{\|\boldsymbol{v}_S\|_{h_S}} \geq \frac{(\nabla_w \cdot \boldsymbol{v}^*, q_S)_{\Omega_S}}{\|\boldsymbol{v}^*\|_{h_S}} \geq \frac{(\nabla \cdot \boldsymbol{v}, q_S)_{\Omega_S}}{\max\{1, C_2\} \|\boldsymbol{v}\|_{1,\Omega_S}} \geq \frac{C_1}{\max\{1, C_2\}} \|q_S\|_{\Omega_S}.$$

The proof is complete. □

Following Lemmas 3.2-3.4, Remark 3.2 and the analysis in Section 2, we can derive the following theorem.

Theorem 3.1. *If $g \in L^2(\Gamma_{S_2})$, the WG scheme* (3.6) *has a unique solution $(\boldsymbol{u}_S^h, p_S^h, p_D^h) \in \boldsymbol{U}_{h_S}^0 \times P_{h_S}^0 \times P_{h_D}^0$.*

4. Error Inequality

An error inequality for the WG scheme (3.6) will be given in this section. Let $(\boldsymbol{u}_S^h, p_S^h, p_D^h) \in \boldsymbol{U}_{h_S}^0 \times P_{h_S}^0 \times P_{h_D}^0$ be the solution of (3.6), with $\boldsymbol{u}_S^h = \{\boldsymbol{u}_0^h, \boldsymbol{u}_b^h\}$ and $p_D^h = \{p_0^h, p_b^h\}$ and let

$$(\boldsymbol{u}_S, p_S, p_D) \in \left(U_S^0 \times L_0^2(\Omega_S) \times P_D^0\right) \cap \left([H^2(\Omega_S)]^2 \times H^1(\Omega_S) \times H^2(\Omega_D)\right) \tag{4.1}$$

be the solution of (1.1)-(1.12). With the symbols above, we define the following error functions:

$$\begin{aligned} \boldsymbol{e}_S &= \{\boldsymbol{e}_{S_0}, \boldsymbol{e}_{S_b}\} = \left\{\boldsymbol{Q}_0 \boldsymbol{u}_S - \boldsymbol{u}_0^h, \boldsymbol{Q}_b \boldsymbol{u}_S - \boldsymbol{u}_b^h\right\} = \boldsymbol{Q}_h \boldsymbol{u}_S - \boldsymbol{u}_S^h, \\ e_D &= \{e_{D_0}, e_{D_b}\} = \left\{Q_0 p_D - p_0^h, Q_b p_D - p_b^h\right\} = Q_h p_D - p_D^h. \end{aligned} \tag{4.2}$$

Lemma 4.1. *Given $\boldsymbol{w} = \{\boldsymbol{w}_0, \boldsymbol{w}_b\} \in \boldsymbol{U}_{h_S}$ and $q = \{q_0, q_b\} \in U_{h_D}$, we have*

$$\begin{aligned} &\left(e_w(\boldsymbol{w}), \boldsymbol{\omega}\right)_T = \left(e(\boldsymbol{w}_0), \boldsymbol{\omega}\right)_T + \langle \boldsymbol{w}_b - \boldsymbol{w}_0, \boldsymbol{\omega}\boldsymbol{n}\rangle_{\partial T} && \text{for all } T \in \mathcal{T}_{h_S} \text{ and all } \boldsymbol{\omega} \in \overline{\boldsymbol{M}}_{h_S}, \\ &(\nabla_w q, \boldsymbol{\phi})_T = (\nabla q, \boldsymbol{\phi})_T + \langle \boldsymbol{q}_b - \boldsymbol{q}_0, \boldsymbol{\phi} \cdot \boldsymbol{n}\rangle_{\partial T} && \text{for all } T \in \mathcal{T}_{h_D} \text{ and all } \boldsymbol{\phi} \in \boldsymbol{U}_{h_D}, \\ &(\nabla_w \cdot \boldsymbol{w}, v)_T = (\nabla \cdot \boldsymbol{w}_0, v)_T + \langle \boldsymbol{w}_b - \boldsymbol{w}_0, v\boldsymbol{n}\rangle_{\partial T} && \text{for all } T \in \mathcal{T}_{h_S} \text{ and all } v \in P_{h_S}. \end{aligned}$$

Proof. From Definitions 3.1-3.2, Remark 3.1 and the integration by parts, we get the conclusions. □

Lemma 4.2. *Let $\boldsymbol{e}_S$ and e_D be defined in* (4.2), *then for any $(\boldsymbol{v}_S, q_D) \in \mathcal{S}_h$ with $\boldsymbol{v}_S = \{\boldsymbol{v}_0, \boldsymbol{v}_b\}$ and $q_D = \{q_0, q_b\}$, we have*

$$\begin{aligned} &a_S^h\left(\boldsymbol{e}_S, \boldsymbol{v}_S - \boldsymbol{u}_S^h\right) + a_D^h\left(e_D, q_D - p_D^h\right) \\ \leq\, & J(\boldsymbol{v}_S) - J\left(\boldsymbol{u}_S^h\right) + \mathcal{E}\left(\boldsymbol{v}_S - \boldsymbol{u}_S^h, q_D - p_D^h\right) \\ &+ \mathcal{L}\left(\boldsymbol{v}_S - \boldsymbol{u}_S^h, q_D - p_D^h\right) + \mathcal{S}\left(\boldsymbol{v}_S - \boldsymbol{u}_S^h, q_D - p_D^h\right), \end{aligned} \tag{4.3}$$

where

$$
\begin{aligned}
J(\boldsymbol{v}_S) &= j^h(v_{S,t}) + \int_{\Gamma_{S_2}} \sigma_t v_{b,t}\, \mathrm{d}s,\\
\mathscr{E}(\boldsymbol{v}_S, q_D) &= 2\nu \sum_{T\in\mathcal{T}_{h_S}} \big\langle \boldsymbol{v}_0 - \boldsymbol{v}_b, (e(\boldsymbol{u}_S) - \mathbb{Q}_h e(\boldsymbol{u}_S))\boldsymbol{n}\big\rangle_{\partial T}\\
&\quad - \sum_{T\in\mathcal{T}_{h_S}} \big\langle \boldsymbol{v}_0 - \boldsymbol{v}_b, (p_S - \mathbb{Q}_h p_S)\boldsymbol{n}\big\rangle_{\partial T}\\
&\quad - \sum_{T\in\mathcal{T}_{h_D}} \frac{1}{\mu}\big(K(\nabla p_D - \mathbb{Q}_h(\nabla p_D)), \nabla_w q_D\big)_T \qquad (4.4)\\
&\quad + \sum_{T\in\mathcal{T}_{h_D}} \frac{1}{\mu}\big\langle q_0 - q_b, (K\nabla p_D - \mathbb{Q}_h(K\nabla p_D))\cdot \boldsymbol{n}\big\rangle_{\partial T},\\
\mathscr{L}(\boldsymbol{v}_S, q_D) &= -B^h(\boldsymbol{e}_S, \boldsymbol{v}_S; e_D, q_D),\\
\mathscr{S}(\boldsymbol{v}_S, q_D) &= 2\nu s_S(\boldsymbol{Q}_h \boldsymbol{u}_S, \boldsymbol{v}_S) + \frac{1}{\mu} m_K s_D(Q_h p_D, q_D).
\end{aligned}
$$

Proof. Given $q_D = \{q_0, q_b\} \in P^0_{h_D}$ and $\boldsymbol{v}_S = \{\boldsymbol{v}_0, \boldsymbol{v}_b\} \in \boldsymbol{U}^0_{h_S}$, it follows from (2.4) and (2.5) that

$$
-\big(\nabla\cdot(2\nu e(\boldsymbol{u}_S)), \boldsymbol{v}_0\big)_{\Omega_S} + (\nabla p_S, \boldsymbol{v}_0)_{\Omega_S} + (\nabla\cdot \boldsymbol{u}_D, q_0)_{\Omega_D} = (\boldsymbol{f}_S, \boldsymbol{v}_0)_{\Omega_S} + (f_D, q_0)_{\Omega_D}. \qquad (4.5)
$$

From Lemmas 3.1 and 4.1, we obtain

$$
\begin{aligned}
-\big(\nabla\cdot(2\nu e(\boldsymbol{u}_S)), \boldsymbol{v}_0\big)_{\Omega_S} &= 2\nu\big(e(\boldsymbol{u}_S), e(\boldsymbol{v}_0)\big)_{\Omega_S} - 2\nu \sum_{T\in\mathcal{T}_{h_S}} \big\langle e(\boldsymbol{u}_S)\boldsymbol{n}, \boldsymbol{v}_0\big\rangle_{\partial T}\\
&= 2\nu\big(e_w(\boldsymbol{Q}_h\boldsymbol{u}_S), e_w(\boldsymbol{v}_S)\big)_{\Omega_S} - 2\nu\big\langle e(\boldsymbol{u}_S)\boldsymbol{n}, \boldsymbol{v}_b\big\rangle_{\partial\Omega_S}\\
&\quad + 2\nu \sum_{T\in\mathcal{T}_{h_S}} \big\langle \boldsymbol{v}_0 - \boldsymbol{v}_b, (\mathbb{Q}_h e(\boldsymbol{u}_S) - e(\boldsymbol{u}_S))\boldsymbol{n}\big\rangle_{\partial T}, \qquad (4.6)\\
(\nabla p_S, \boldsymbol{v}_0)_{\Omega_S} &= -(p_S, \nabla\cdot \boldsymbol{v}_0)_{\Omega_S} + \sum_{T\in\mathcal{T}_{h_S}} \langle \boldsymbol{v}_0, p_S\boldsymbol{n}\rangle_{\partial T}\\
&= -(\mathbb{Q}_h p_S, \nabla_w\cdot \boldsymbol{v}_S)_{\Omega_S} + \langle \boldsymbol{v}_b, p_S \boldsymbol{n}_S\rangle_{\partial\Omega_S}\\
&\quad + \sum_{T\in\mathcal{T}_{h_S}} \big\langle \boldsymbol{v}_0 - \boldsymbol{v}_b, (p_S - \mathbb{Q}_h p_S)\boldsymbol{n}\big\rangle_{\partial T}, \qquad (4.7)\\
(\nabla\cdot \boldsymbol{u}_D, q_0)_{\Omega_D} &= -(\boldsymbol{u}_D, \nabla q_0)_{\Omega_D} + \sum_{T\in\mathcal{T}_{h_D}} \langle \boldsymbol{u}_D\cdot\boldsymbol{n}, q_0\rangle_{\partial T}\\
&= \frac{1}{\mu}\big(K(\nabla p_D - \mathbb{Q}_h\nabla p_D), \nabla_w q_D\big)_{\Omega_D} + \frac{1}{\mu}(K\nabla_w e_D, \nabla_w q_D)_{\Omega_D}\\
&\quad + \sum_{T\in\mathcal{T}_{h_D}} \frac{1}{\mu}\big\langle q_0 - q_b, (\mathbb{Q}_h(K\nabla p_D) - K\nabla p_D)\cdot\boldsymbol{n}\big\rangle_{\partial T}
\end{aligned}
$$

$$+\frac{1}{\mu}\left(K\nabla_w p_D^h, \nabla_w q_D\right)_{\Omega_D} + \langle \boldsymbol{u}_D\cdot\boldsymbol{n}_D, q_b\rangle_{\partial\Omega_D}. \tag{4.8}$$

With the boundary conditions in (1.1)-(1.12), we derive

$$\begin{aligned}&-2\nu\big\langle e(\boldsymbol{u}_S)\boldsymbol{n}_S, \boldsymbol{v}_b\big\rangle_{\partial\Omega_S} + \langle \boldsymbol{v}_b, p_S\boldsymbol{n}_S\rangle_{\partial\Omega_S} + \langle \boldsymbol{u}_D\cdot\boldsymbol{n}_D, q_b\rangle_{\partial\Omega_D}\\ &= B^h(\boldsymbol{u}_S, \boldsymbol{v}_S; p_D, q_D) - \int_{\Gamma_{S_2}} \sigma_t v_{b,t}\,\mathrm{d}s.\end{aligned} \tag{4.9}$$

Replacing $\boldsymbol{v}_0$ and q_0 by $\boldsymbol{v}_0-\boldsymbol{u}_0^h$ and $q_0-p_0^h$ respectively in (4.5), then subtracting the WG scheme (3.6) from (4.5), and with (4.6)-(4.9), we get (4.3). $\square$

5. Error Estimate in h-Norm

We will estimate $\|\boldsymbol{e}_S\|_{h_S}$ and $\|e_D\|_{h_D}$ in this section. Firstly, we estimate the terms in the error inequality (4.3) separately.

Lemma 5.1. *Suppose the permeability tensor $K\in[W^{1,\infty}(\Omega_D)]^{2\times 2}$, cf.* [1, Section 3.1] *for the definition of $W^{1,\infty}$. Then for $\mathscr{E}(\boldsymbol{v}_S, q_D)$ defined in* (4.4)*, we have*

$$\mathscr{E}(\boldsymbol{v}_S, q_D) \le Ch\left(\|\boldsymbol{u}_S\|_{2,\Omega_S} + \|p_S\|_{1,\Omega_S}\right)\|\boldsymbol{v}_S\|_{h_S} + Ch\|p_D\|_{2,\Omega_D}.$$

Proof. From [24, Lemma -4.1], we obtain

$$\|v-\mathbb{Q}_h v\|_T^2 + h_T^2\|\nabla(v-\mathbb{Q}_h v)\|_T^2 \le Ch^{2s}\|v\|_{s,T}^2 \tag{5.1}$$

for any $v\in H^s(\Omega_i)$, $i\in\{S,D\}$, $T\in\mathscr{T}_{h_i}$ and $s\in\{1,2\}$. Due to Lemma 3.3 and (5.1), we have

$$h_T\|v-\mathbb{Q}_h v\|_{\partial T}^2 \le C\left(\|v-\mathbb{Q}_h v\|_T^2 + h_T^2\|\nabla(v-\mathbb{Q}_h v)\|_T^2\right) \le Ch^{2s}\|v\|_{s,T}^2. \tag{5.2}$$

By using (4.1), (5.1), and (5.2), we get

$$\begin{aligned}&h_T\|e(\boldsymbol{u}_S)-\mathbb{Q}_h e(\boldsymbol{u}_S)\|_{\partial T}^2 \le Ch^2\|e(\boldsymbol{u}_S)\|_{1,T}^2 \le Ch^2\|\boldsymbol{u}_S\|_{2,T}^2,\\ &h_T\|p_S-\mathbb{Q}_h p_S\|_{\partial T}^2 \le Ch^2\|p_S\|_{1,T}^2,\\ &\|\nabla p_D-\mathbb{Q}_h(\nabla p_D)\|_T^2 \le Ch^2\|\nabla p_D\|_{1,T}^2 \le Ch^2\|p_D\|_{2,T}^2,\\ &h_T\|K\nabla p_D-\mathbb{Q}_h(K\nabla p_D)\|_{\partial T}^2 \le Ch^2\|K\nabla p_D\|_{1,T}^2 \le Ch^2\|p_D\|_{2,T}^2,\end{aligned} \tag{5.3}$$

where

$$\|K\nabla p_D\|_{1,T} \le \|K\|_{W^{1,\infty}(T)}\|\nabla p_D\|_{1,T} \le C\|p_D\|_{2,T}$$

is used in the last line. Then, in view of the Cauchy-Schwarz inequality and (5.3), we arrive at

$$\begin{aligned}&2\nu\sum_{T\in\mathscr{T}_{h_S}}\big\langle \boldsymbol{v}_0-\boldsymbol{v}_b, (e(\boldsymbol{u}_S)-\mathbb{Q}_h e(\boldsymbol{u}_S))\boldsymbol{n}\big\rangle_{\partial T}\\ &\le 2\nu\bigg(\sum_{T\in\mathscr{T}_{h_S}} h_T^{-1}\|\boldsymbol{v}_0-\boldsymbol{v}_b\|_{\partial T}^2\bigg)^{1/2}\bigg(\sum_{T\in\mathscr{T}_{h_S}} h_T\|e(\boldsymbol{u}_S)-\mathbb{Q}_h e(\boldsymbol{u}_S)\|_{\partial T}^2\bigg)^{1/2}\\ &\le Ch\|\boldsymbol{u}_S\|_{2,\Omega_S}\|\boldsymbol{v}_S\|_{h_S},\end{aligned} \tag{5.4}$$

$$
\begin{aligned}
&-\sum_{T\in\mathcal{T}_{h_S}}\big\langle \boldsymbol{v}_0-\boldsymbol{v}_b,(p_S-\mathbb{Q}_hp_S)\boldsymbol{n}\big\rangle_{\partial T}\\
\le&\Big(\sum_{T\in\mathcal{T}_{h_S}}h_T^{-1}\|\boldsymbol{v}_0-\boldsymbol{v}_b\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_{h_S}}h_T\|p_S-\mathbb{Q}_hp_S\|_{\partial T}^2\Big)^{1/2}\\
\le& Ch\|p_S\|_{2,\Omega_S}\|\boldsymbol{v}_S\|_{h_S}, \qquad (5.5)\\
&-\sum_{T\in\mathcal{T}_{h_D}}\frac{1}{\mu}\big(K(\nabla p_D-\mathbb{Q}_h(\nabla p_D)),\nabla_w q_D\big)_T\\
\le& C\Big(\sum_{T\in\mathcal{T}_{h_D}}\|\nabla p_D-\mathbb{Q}_h(\nabla p_D)\|_T^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_{h_D}}\|\nabla_w q_D\|_T^2\Big)^{1/2}\\
\le& Ch\|p_D\|_{2,\Omega_D}\|q_D\|_{h_D}, \qquad (5.6)\\
&\sum_{T\in\mathcal{T}_{h_D}}\frac{1}{\mu}\big\langle q_0-q_b,(K\nabla p_D-\mathbb{Q}_h(K\nabla p_D))\cdot\boldsymbol{n}\big\rangle_{\partial T}\\
\le&\frac{1}{\mu}\Big(\sum_{T\in\mathcal{T}_{h_D}}h_T^{-1}\|q_0-q_b\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_{h_D}}h_T\|K\nabla p_D-\mathbb{Q}_h(K\nabla p_D)\|_{\partial T}^2\Big)^{1/2}\\
\le& Ch\|p_D\|_{2,\Omega_D}\|q_D\|_{h_D}. \qquad (5.7)
\end{aligned}
$$

Combining (5.4)-(5.7), we derive the conclusion. □

Remark 5.1. When we discuss the well-posedness for the variational inequality (2.12) and the WG scheme (3.6), we only need to assume the permeability tensor $K=K(\boldsymbol{x})\,(\boldsymbol{x}\in\Omega_D)$ is uniformly positive definite and bounded, without additional assumptions to the regularity of K. However, when we try to obtain the error estimates for the WG approximation, the regularity of K affects the convergence orders. For any $T\in\mathcal{T}_{h_D}$, if $K|_T\notin[P_0(T)]^{2\times2}$, $\mathbb{Q}_h(K\nabla p_D)\neq K\mathbb{Q}_h(\nabla p_D)$. With the Cauchy-Schwarz inequality, we can obtain

$$\sum_{T\in\mathcal{T}_{h_D}}\|K\nabla p_D-\mathbb{Q}_h(K\nabla p_D)\|_T\le Ch\|K\nabla p_D\|_{1,\Omega_D}\le Ch\|K\|_{W^{1,\infty},\Omega_D}\|p_D\|_{2,\Omega_D}.$$

This is the reason that we assume $K\in[W^{1,\infty}(\Omega_D)]^{2\times2}$ in Lemma 5.1.

Lemma 5.2. *For $\mathcal{S}(\boldsymbol{v}_S,q_D)$ defined in* (4.4)*, we have*

$$\mathcal{S}(\boldsymbol{v}_S,q_D)\le Ch\|\boldsymbol{u}_S\|_{2,\Omega_S}\|\boldsymbol{v}_S\|_{h_S}+Ch\|p_D\|_{2,\Omega_D}\|q_D\|_{h_D}.$$

Proof. Due to (3.4), (4.2) and Lemma 3.3, similar to (5.2), we arrive at

$$
\begin{aligned}
2\nu s_S(\mathbf{Q}_h\boldsymbol{u}_S,\boldsymbol{v}_S)\le& 2\nu\Big(\sum_{T\in\mathcal{T}_{h_S}}h_T^{-1}\|\mathbf{Q}_0\boldsymbol{u}_S-\boldsymbol{u}_S\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_{h_S}}h_T^{-1}\|\boldsymbol{v}_0-\boldsymbol{v}_b\|_{\partial T}^2\Big)^{1/2}\\
\le& Ch\|\boldsymbol{u}_S\|_{2,\Omega_S}\|\boldsymbol{v}_S\|_{h_S}, \qquad (5.8)
\end{aligned}
$$

$$\begin{aligned}\frac{1}{\mu}m_K s_D(Q_h p_D, q_D) &\leq \frac{1}{\mu}m_K\Big(\sum_{T\in\mathcal{T}_{h_D}} h_T^{-1}\|Q_0 p_D - p_D\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_{h_D}} h_T^{-1}\|q_0 - q_b\|_{\partial T}^2\Big)^{1/2}\\ &\leq Ch\|p_D\|_{2,\Omega_D}\|q_D\|_{h_D}. \end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9), we get the conclusion. □

Lemma 5.3. *If $g \in L^2(\Gamma_{S_2})$ and $\boldsymbol{u}_S \in [H^2(\Gamma_{S_2})]^2$, then for $J(\cdot)$ defined in* (4.4)*, we have*

$$J(\boldsymbol{Q}_h\boldsymbol{u}_S) - J\left(\boldsymbol{u}_S^h\right) \leq Ch^2\|\boldsymbol{u}_S\|_{2,\Gamma_{S_2}}.$$

Proof. From (1.9), we find

$$\begin{aligned} J\left(\boldsymbol{u}_S^h\right) &= \int_{\Gamma_{S_2}}\left(g|u_{b_h,t}| + \sigma_t u_{b_h,t}\right)\mathrm{d}s \geq 0,\\ J(\boldsymbol{u}_S) &= \int_{\Gamma_{S_2}}\left(g|u_{S,t}| + \sigma_t u_{S,t}\right)\mathrm{d}s = 0. \end{aligned}$$

With the above relations, $|\sigma_t| \leq g$ in (1.9) and the properties of absolute value, we get

$$\begin{aligned} J(\boldsymbol{Q}_h\boldsymbol{u}_S) - J\left(\boldsymbol{u}_S^h\right) &\leq J(\boldsymbol{Q}_h\boldsymbol{u}_S) - J(\boldsymbol{u}_S)\\ &\leq \int_{\Gamma_{S_2}}\big(g|\boldsymbol{Q}_b\boldsymbol{u}_S - \boldsymbol{u}_S| + |\sigma_t(\boldsymbol{Q}_b\boldsymbol{u}_S - \boldsymbol{u}_S)|\big)\mathrm{d}s\\ &\leq 2\int_{\Gamma_{S_2}} g|\boldsymbol{Q}_b\boldsymbol{u}_S - \boldsymbol{u}_S|\,\mathrm{d}s\\ &\leq 2\|g\|_{\Gamma_{S_2}}\|\boldsymbol{Q}_b\boldsymbol{u}_S - \boldsymbol{u}_S\|_{\Gamma_{S_2}}\\ &\leq Ch^2\|\boldsymbol{u}_S\|_{2,\Gamma_{S_2}}. \end{aligned}$$

The proof is complete. □

Theorem 5.1. *Let $\boldsymbol{e}_S$ and e_D be defined in* (4.2) *and $K \in [W^{1,\infty}(\Omega_D)]^{2\times 2}$. Under the assumptions of Lemma* 5.3*, we have*

$$\|\boldsymbol{e}_S\|_{h_S} + \|e_D\|_{h_D} \leq Ch.$$

Proof. Taking $\boldsymbol{v}_S = \boldsymbol{Q}_h\boldsymbol{u}_S$ and $q_D = Q_h p_D$ in the error inequality (4.3), we get

$$a_S(\boldsymbol{e}_S,\boldsymbol{e}_S) + a_D(e_D,e_D) \leq J(\boldsymbol{Q}_h\boldsymbol{u}_S) - J\left(\boldsymbol{u}_S^h\right) + \mathscr{E}(\boldsymbol{e}_S,e_D) + \mathscr{L}(\boldsymbol{e}_S,e_D) + \mathscr{S}(\boldsymbol{e}_S,e_D). \tag{5.10}$$

It follows from (4.4) and Lemmas 5.1-5.3, that

$$\begin{aligned} &J(\boldsymbol{Q}_h\boldsymbol{u}_S) - J\left(\boldsymbol{u}_S^h\right) \leq Ch^2,\\ &\mathscr{E}(\boldsymbol{e}_S,e_D) \leq Ch\|\boldsymbol{e}_S\|_{h_S} + Ch\|e_D\|_{h_D},\\ &\mathscr{L}(\boldsymbol{e}_S,e_D) = -\kappa\langle e_{b_S,t}, e_{b_S,t}\rangle_\Gamma \leq 0,\\ &\mathscr{S}(\boldsymbol{e}_S,e_D) \leq Ch\|\boldsymbol{e}_S\|_{h_S} + Ch\|e_D\|_{h_D}. \end{aligned} \tag{5.11}$$

The relations (2.26), (5.10), (5.11) and the Young's inequality give

$$\|\boldsymbol{e}_S\|_{h_S}^2 + \|e_D\|_{h_D}^2 \le Ch^2, \quad h \le 1,$$

which yields

$$\|\boldsymbol{e}_S\|_{h_S} + \|e_D\|_{h_D} \le \sqrt{2\left(\|\boldsymbol{e}_S\|_{h_S}^2 + \|e_D\|_{h_D}^2\right)} \le Ch.$$

The proof is complete. □

6. Numerical Examples

This section gives two numerical examples for the approximation of the problem (1.1)-(1.12). Here, we use the WG scheme (3.6) with the uniform triangular partitions. In all examples, we set $\kappa = 1$ and consider sufficiently smooth parameters $(\nu, \mu, K, \boldsymbol{f}_S, f_D, g)$.

Example 6.1. Let $\Gamma = (0,1) \times \{1\}$, $\Gamma_{S_1} = \{0,1\} \times (0,1)$, $\Gamma_{S_2} = (0,1) \times \{0\}$ and $\Gamma_D = \partial\Omega_D \setminus \overline{\Gamma}$ (shown in Fig. 2). We take the exact solution

$$\boldsymbol{u}_S = \begin{pmatrix} x^3(1-x)^3 y^2(1-y)^2(2y-1) \\ -y^3(1-y)^3 x^2(1-x)^3(2x-1) \end{pmatrix}, \quad p_S = (2x-1)(y-1),$$

$$\boldsymbol{u}_D = \begin{pmatrix} \frac{1}{\mu}(2x-1)(y-1)^2(y-2) \\ \frac{1}{\mu}x(x-1)(y-1)(3y-5) \end{pmatrix}, \qquad p_D = x(1-x)(y-2)(y-1)^2$$

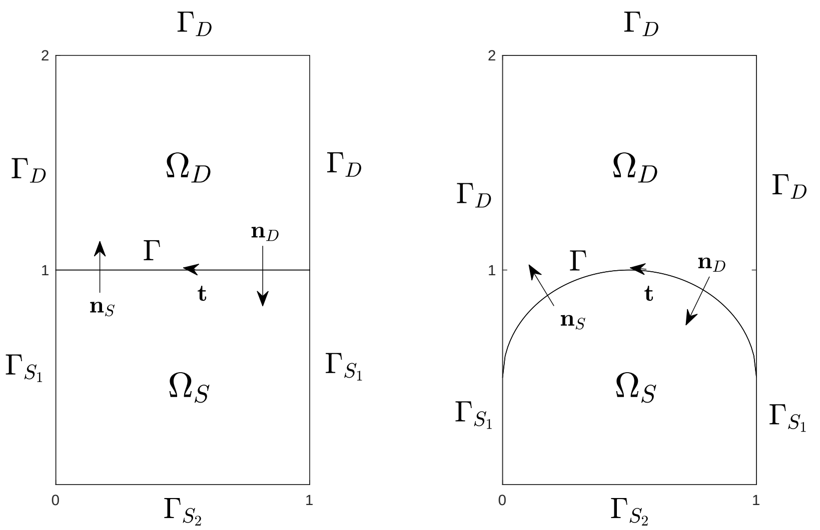

Figure 2: Domains for Example 6.1 (left) and Example 6.2 (right).

with $\sigma_t(\boldsymbol{u}_S) = 0$ on Γ_{S_2}. Here, $\mu = \nu = 1$ and the permeability tensor $K = \mathbb{I} \in \mathbb{R}^{2\times 2}$ is the identity tensor. And we choose the barrier function as

$$g(x,y) = 0 \quad \text{for all} \quad (x,y) \in \Gamma_{S_2}.$$

The corresponding $\boldsymbol{f}_S$ and f_D are calculated from the exact solution.

In Table 1, we list the errors and the convergence orders of the WG method. It can be found that the estimates for $\|\boldsymbol{e}_S\|_{h_S}$ and $\|e_D\|_{h_D}$ are $\mathcal{O}(h)$, which agrees with the conclusions in Theorem 5.1.

Table 1: Errors and orders for Example 6.1.

$h/\sqrt{2}$	$\|\boldsymbol{e}_S\|_{h_S}$	Order	$\|e_D\|_{h_D}$	Order	$\|\boldsymbol{e}_{S_0}\|_{\Omega_S}$	Order	$\|e_{D_0}\|_{\Omega_D}$	Order
8	4.236e-02	-	2.777e-02	-	2.920e-03	-	1.962e-03	-
16	2.116e-02	1.00	1.388e-02	1.00	7.281e-04	2.00	4.924e-04	1.99
32	1.057e-02	1.00	6.940e-03	1.00	1.813e-04	2.01	1.232e-04	2.00
64	5.281e-03	1.00	3.470e-03	1.00	4.524e-05	2.00	3.081e-05	2.00
128	2.640e-03	1.00	1.735e-03	1.00	1.130e-05	2.00	7.704e-06	2.00

Example 6.2. Let $\Gamma = \{(x,y) : y = 0.5 + \sqrt{0.25-(x-0.5)^2}, 0 < x < 1\}$, $\Gamma_{S_1} = \{(x,y) : x = 0 \text{ or } x = 1, 0 < y < 0.5\}$, $\Gamma_{S_2} = \{(x,y) : y = 0, 0 < x < 1\}$ and $\Gamma_D = \{(x,y) : x = 0 \text{ or } x = 1, 0.5 < y < 2\} \cup \{(x,y) : y = 2, 0 < x < 1\}$ (shown in Fig. 2). Moreover, we choose the right hand sides, the barrier function and the permeability tensor as

$$\begin{aligned}
&\boldsymbol{f}_S(x,y) = \epsilon_{f_S} * (1,1)^\intercal && \text{for all} \quad (x,y) \in \Omega_S,\\
&f_D(x,y) = \epsilon_{f_D} && \text{for all} \quad (x,y) \in \Omega_D,\\
&g(x,y) = \epsilon_g && \text{for all} \quad (x,y) \in \Gamma_{S_2},\\
&K(x,y) = m_K \mathbb{I} && \text{for all} \quad (x,y) \in \Omega_D
\end{aligned}$$

with $\epsilon_{f_S}, \epsilon_{f_D}, \epsilon_g \in (0,\infty)$ being sufficiently small, which can be seen as the perturbations to $(\boldsymbol{f}_S, f_D, g)$. From [21, Table 4.20], the viscosity values mostly range from 10^{-6} $Pa\cdot s$ to 10^2 $Pa \cdot s$. And from [19, Tables A.1, A.2], the permeability values mostly range from 10^{-15} m^2 to 10^{-7} m^2. Then we choose

$$\begin{aligned}
&\mu, \nu \in \left\{10^{-6}, 10^{-4}, 10^{-2}, 1, 10^2\right\},\\
&m_K \in \left\{10^{-15}, 10^{-13}, 10^{-11}, 10^{-9}, 10^{-7}\right\}.
\end{aligned}$$

Here, the exact solution is unknown. Set $\epsilon_{f_S} = \epsilon_{f_D} = \epsilon_g = 10^{-8}$ and $h = \sqrt{2}/128$. Then, we consider the following cases:

(I) $\mu = 10^{-6}$ and $m_K = 10^{-7}$ with different $\nu \in \{10^{-6}, 10^{-4}, 10^{-2}, 1, 10^2\}$;

(II) $\nu = 10^2$ and $m_K = 10^{-7}$ with different $\mu \in \{10^{-6}, 10^{-4}, 10^{-2}, 1, 10^2\}$;

(III) $\nu=10^2$ and $\mu=10^{-6}$ with different $m_K\in\{10^{-15},10^{-13},10^{-11},10^{-9},10^{-7}\}$.

With the parameters above, we can get the WG solution $(\boldsymbol{u}_S^h,p_S^h,p_D^h)$ from (3.6). Since the exact solution is unknown, similar to [33, Example 4.4], we define the following errors:

$$\begin{aligned}
Err_0 &= \left\|\nabla_w\cdot\boldsymbol{u}_S^h\right\|_{\Omega_S},\\
Err_1 &= \left\|\boldsymbol{u}_S^h\cdot\boldsymbol{n}_S+\frac{1}{\mu}K\nabla_w p_D^h\cdot\boldsymbol{n}_S\right\|_{\Gamma},\\
Err_2 &= \left\|2\nu e_w\left(\boldsymbol{u}_S^h\right)\boldsymbol{n}_S\cdot\boldsymbol{n}_S-p_S^h+p_D^h\right\|_{\Gamma},\\
Err_3 &= \left\|2\nu e_w\left(\boldsymbol{u}_S^h\right)\boldsymbol{n}_S\cdot\boldsymbol{t}+\kappa\boldsymbol{u}_S^h\cdot\boldsymbol{t}\right\|_{\Gamma}.
\end{aligned}$$

In fact, Err_0,Err_1,Err_2, and Err_3 are the approximations to (1.5), (1.10), (1.11) and (1.12) respectively. To study the robustness of the WG method with respect to ν,μ and m_K, we list the errors for cases I–III in Tables 2-4 respectively.

Due to Lemma 2.4 and the definitions of $a_S^h(\cdot,\cdot)$ and $a_D^h(\cdot,\cdot)$ in (3.5), the singularity of the WG method is governed by $\alpha=\min\{2\nu,m_K/\mu\}$, which determines the lower bound of

Table 2: Errors for case I of Example 6.2.

ν	Err_0	Err_1	Err_2	Err_3
10^{-6}	3.223e-06	6.319e-09	8.329e-09	4.627e-09
10^{-4}	2.123e-07	2.445e-09	2.175e-09	3.472e-09
10^{-2}	2.484e-09	3.739e-10	2.033e-09	2.204e-09
1	3.676e-11	3.150e-11	1.008e-09	7.648e-10
10^2	3.819e-13	2.850e-11	8.478e-10	9.352e-10

Table 3: Errors for case II of Example 6.2.

μ	Err_0	Err_1	Err_2	Err_3
10^2	3.409e-07	1.498e-09	9.405e-03	1.394e-05
1	2.217e-08	4.170e-10	5.989e-04	2.700e-06
10^{-2}	5.997e-10	9.322e-11	1.344e-05	4.169e-07
10^{-4}	3.351e-11	2.923e-11	4.752e-07	2.402e-08
10^{-6}	3.819e-13	2.850e-11	8.478e-10	9.352e-10

Table 4: Errors for case III of Example 6.2.

m_K	Err_0	Err_1	Err_2	Err_3
10^{-15}	3.409e-07	1.498e-09	9.405e-03	1.394e-05
10^{-13}	2.217e-08	4.170e-10	5.989e-04	2.700e-06
10^{-11}	5.997e-10	9.322e-11	1.344e-05	4.169e-07
10^{-9}	3.351e-11	2.923e-11	4.752e-07	2.402e-08
10^{-7}	3.819e-13	2.850e-11	8.478e-10	9.352e-10

the eigenvalues for the WG matrix. If α is too small, the WG method will be singular, then the obtained WG solution is not accurate. In other words, the bigger α is, the more robust the WG method is. So, we will analyze the numerical results in Tables 2-4 according to α. For case I, we have $\alpha = 2\nu$ when $\nu \leq 10^{-2}$ and $\alpha = 10^{-1}$ when $\nu \geq 1$. From Table 2, it can be found that the errors $Err_i, i \in \{0,1,2,3\}$ become smaller as ν becomes bigger for fixed $(\mu, m_K, \boldsymbol{f}_S, f_D, g)$. For case II, we have $\alpha = 10^{-7}/\mu$. Thus, as μ becomes smaller, the errors Err_i, $i \in \{0,1,2,3\}$ become smaller for fixed $(\nu, m_K, \boldsymbol{f}_S, f_D, g)$, see Table 3. For case III, we have $\alpha = m_K \cdot 10^6$. From Table 4, it is easy to see that the errors Err_i, $i \in \{0,1,2,3\}$ become smaller as m_K becomes bigger for fixed $(\nu, \mu, \boldsymbol{f}_S, f_D, g)$. In summary, the WG method (3.6) is robust when ν is bigger, μ is smaller and m_K is bigger.

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