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Finite Genus Solutions to a Hierarchy of Integrable Semi-Discrete Equations

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Abstract. Resorting to the discrete zero-curvature equation and the Lenard recursion equations, a hierarchy of integrable semi-discrete nonlinear evolution equations is derived from a 3×3 matrix spectral problem with three potentials. Based on the characteristic polynomial of the Lax matrix for the hierarchy, a trigonal curve is introduced, and the properties of the corresponding three-sheeted Riemann surface are studied, including the genus, three kinds of Abelian differentials, Riemann theta functions. The asymptotic properties of the Baker-Akhiezer function and fundamental meromorphic functions defined on the trigonal curve are analyzed with the established theory of trigonal curves. As a result, finite genus solutions of the whole integrable semi-discrete nonlinear evolution hierarchy are obtained.

AMS subject classifications: 37K10, 37K20, 14H42, 37K40

Key words: Integrable semi-discrete nonlinear evolution equation, trigonal curve, Baker-Akhiezer function, finite genus solution.

1. Introduction

Finite genus solutions of soliton equations have been a concerning topic over the past few decades, which reveal the inherent structure mechanism of solitons and describe the quasi-periodic behavior of nonlinear phenomena. Since the early 1970s, various methods have been developed to solve soliton equations, such as the inverse scattering method, Darboux transformation, Riemann-Hilbert approach and algebro-geometric methods — cf. [1, 6, 16–19, 27, 29, 30, 40] and references therein. Among these methods, the algebro-geometric method is a powerful tool to construct finite genus solutions of soliton equations associated with 2×2 matrix spectral problems based on the theory of hyperelliptic curves. It has been successfully applied to the KdV, nonlinear Schrödinger, mKdV, sine-Gordon, Toda lattice, and Ablowitz-Ladik equations and others [2, 3, 5, 8, 9, 12, 14, 15, 24, 28, 31, 34, 36]. The main tools used in this method include the theory of hyperelliptic curves, Riemann

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theta functions, Abel differentials, Abel map, and Abel-Jacobi inversion. When considering higher-order matrix spectral problems, the corresponding algebraic curve becomes non-hyperelliptic, which brings great complexity. Consequently, the investigations of finite genus solutions of soliton equations associated with 3×3 matrix spectral problem are relatively rare compared with 2×2 case. Nonetheless, some progress has been made in [4, 7, 13, 32, 33, 35, 37–39], certain finite genus solutions of the Boussinesq equation related to a third-order differential operator were found as special solutions of the Kadomtsev-Petviashvili equation or by the reduction theory of Riemann theta functions. Dickson *et al*. [10, 11] proposed a framework to yield finite genus solutions of the entire Boussinesq hierarchy based on the classical Burchnall-Chaundy polynomial, the Baker-Akhiezer function and the theory of trigonal curves. Based on this, Geng *et al*. [20] further developed an effective way to introduce algebraic curves associated with higher order matrix spectral problems and applied it to construct finite genus solutions of soliton equation hierarchies related to 3×3 matrix spectral problems, such as the modified Boussinesq, Kaup-Kupershmidt, coupled mKdV hierarchies [20, 21, 23, 26, 41].

In this paper, our main purpose is to derive an integrable hierarchy of semi-discrete nonlinear evolution equations associated with 3×3 matrix spectral problem and construct its finite genus solutions based on the theory of trigonal curves. The first member in the hierarchy is the discrete 3-field system

$$\begin{aligned} u_{n,t} &= -(1-u_n v_n)[u_{n+2}(1-u_{n+1}v_{n+1}) - u_{n+1}w_n], \\ v_{n,t} &= (1-u_n v_n)[v_{n-2}(1-u_{n-1}v_{n-1}) - v_{n-1}w_{n-1}], \\ w_{n,t} &= w_n[u_{n+2}v_n(1-u_{n+1}v_{n+1}) - u_{n+1}v_{n-1}(1-u_n v_n)], \end{aligned} \tag{1.1}$$

where $u_n = u(n,t)$, $u_{n,t} = \partial_t u_n$, $n \in \mathbb{Z}, t \in \mathbb{R}$. The same notation is used for $v(n,t)$ and $w(n,t)$. System (1.1) is reduced to

$$\begin{aligned} u_{n,t} &= u_{n+2}(u_n v_n - 1)(1-u_{n+1}v_{n+1}), \\ v_{n,t} &= v_{n-2}(1-u_n v_n)(1-u_{n-1}v_{n-1}) \end{aligned} \tag{1.2}$$

for $w_n = 0$. In Ref. [22], finite genus solutions of the entire discrete integrable hierarchy of (1.2) are constructed. Due to the addition of one potential function, the properties of the trigonal curve corresponding to system (1.1) at origin become more complicated than the ones for (1.2). Therefore, constructing finite genus solutions to the hierarchy of (1.1) requires more effort. In particular, we need to introduce three Abelian differentials of the third kind to characterize the Baker-Akhiezer function and the meromorphic functions involved.

This paper is organized as follows. In Section 2, we construct the discrete integrable hierarchy from a 3×3 matrix spectral problem with three potentials by resorting to the Lenard recursion equations and zero-curvature equations. Section 3 introduces the stationary Baker-Akhiezer function ψ_3, two meromorphic functions ϕ_{13} and ϕ_{23} carrying the data of the divisor and a trigonal curve of genus $m-1$ with two infinite points and three zero points. The analytic properties of ϕ_{13}, ϕ_{23} and ψ_3 are studied by using the theory of trigonal curves. In Section 4, we present the explicit Riemann theta function representations of