East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-185.301023

Vol. **15**, No. 1, pp. 113-137
February 2025

A Selectively Relaxed Splitting Preconditioning Strategy for the Flux-Limited Multi-Group Radiation Diffusion Equations in Three Dimensions

Xiaoqiang Yue[1], Chenxi Zhang[1], Chunyan Chen[1], Xiaowen Xu[2,*] and Shi Shu[1]

[1]*National Center for Applied Mathematics in Hunan, Key Laboratory of Intelligent Computing & Information Processing of Ministry of Education, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan 411105, China.*
[2]*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100094, China.*

Received 31 July 2023; Accepted (in revised version) 30 October 2023.

Abstract. This article is concerned with a matrix splitting preconditioning technique with two selective relaxations and algebraic multigrid subsolves for $(G+2)\times(G+2)$ block-structured sparse linear systems derived from the three-dimensional flux-limited multi-group radiation diffusion equations, where G is the number of photon energy groups. We introduce an easy-to-implement algebraic selection strategy for the sole contributing parameter, report a spectral analysis and investigate the degree of the minimal polynomial of its left and right preconditioned matrices, and discuss its sequential practical implementation together with the two-level parallelization. Experiments are run with the representative real-world unstructured capsule implosion test cases and it is found that the numerical robustness, computational efficiency and parallel scalability of the proposed preconditioner evaluated on the Tianhe-2A supercomputer with up to 2,816 processor cores are superior to some existing popular monolithic and block preconditioning approaches.

AMS subject classifications: 65F10, 65N55, 65Y05, 65Z05

Key words: Radiation diffusion equation, matrix splitting preconditioner, selective relaxation, algebraic multigrid, parallel computing.

1. Introduction

The numerical simulation of the thermal radiation transport [39] requires solving complicated partial differential equations (PDEs) of parabolic type with highly nonlinear coefficients in a background medium with numerous materials. It is worth noting that the

*Corresponding author. *Email addresses:* yuexq@xtu.edu.cn (X.Q. Yue), xwxu@iapcm.ac.cn (X.W. Xu), shushi@xtu.edu.cn (S. Shu)

thermal radiation transport process occurs in various branches of physics, such as the optical remote sensing, the massive star formation and the inertial confinement fusion experiments. During the past couple of decades, a large amount of research efforts have been devoted to the development of mathematically less complicated and computationally cheaper yet numerically more accurate approximations. Among these approximations, the simplest and the most extensively used one is the flux-limited multi-group radiation diffusion (MGD) equations [21], where the frequency-dependent radiation energy densities are categorized into multiple photon energy (or frequency) groups and, more importantly, are assumed to be equably distributed over the respective frequency ranges.

Traditionally, the adaptive backward Eulerian time-integration is utilized to eliminate the need of severe time-step constraints for the numerical stability. For the resulting nonlinear reaction-diffusion systems, we take advantage of the method of frozen coefficients [30] as the iterative linearization technique, followed by a cell-centered finite volume discretization scheme [11,44] ensuring a local conservation property, however, requiring the solution of a great deal of sparse, ill-conditioned, unsymmetric but positive definite linear systems with the number of degrees of freedom ranging from 10^7 to 10^{11} because of the presence of hydrodynamic instabilities as well as the wave-like propagation characteristics and multiple spatio-temporal scales in exact and approximate solutions, which is computationally expensive, generally accounting for more than eighty percent of the total simulation time. Therefore, this motivates research efforts aimed at the development of robust, accurate and reliable numerical solution algorithms in an efficient and scalable manner.

Despite the reliability and accessibility of sparse direct solvers such as MUMPS [2], PARDISO [42], PaStiX [23], STRUMPACK [19], SuperLU [34], SuperMF [48] and UMFPACK [13] for small systems of linear equations, the memory requirements and difficulties in developing effective massively parallel implementations restrict their scope of practical applications. While on the contrary, the sparse iterative solvers become increasingly attractive to meet the sustaining demand for higher-spatial resolutions, due to their smaller memory utilizations, far easier to implement on parallel computers and higher degrees of parallelism. One of the most powerful sparse iterative solvers is on the strength of specific projections/orthogonalizations onto Krylov subspaces — e.g. Bi-CGSTAB [46], CG [25], GMRES [40] and MINRES [37] as the prominent representatives, whose numerical performance, such as their convergence behaviors, needs to be further boosted via favorable preconditioners. They consistently transform the original system of linear equations into a mathematically equivalent linear system, however, with some more advantageous properties — e.g. smaller (spectral) condition numbers and a more clustered eigenvalue distribution. It is worth highlighting that the two fundamental peculiarities of modern preconditioning approaches are the numerical robustness (with reference to the geometric, physical and discrete parameters and the number of processors) and the implementation scalability — i.e. the setup phase and every iteration step should be scalable in a parallel environment [12]. The numerical robustness is, without doubt, a preemptive requirement to attain a scalable implementation.

Multitudinous scholars and experts at home and abroad have come up with a wide range of resultful preconditioning schemes in an efficient and scalable manner. They can