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A Relaxation Two-Step Newton-Based Matrix Splitting Iteration Method for Generalized Absolute Value Equations Associated with Circular Cones

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Abstract. A relaxation two-step Newton-based matrix splitting (RTNMS) iteration method is proposed to solve the generalized absolute value equations associated with circular cones (CCGAVE). The convergence of the RTNMS iteration method is investigated under suitable conditions. Numerical results illustrate that the RTNMS iteration method is feasible and effective for solving the CCGAVE. Moreover, some sufficient conditions for the unique solvability of the CCGAVE are given.

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Key words: Generalized absolute value equations, circular cone, Newton-based matrix splitting method, convergence analysis.

1. Introduction

The circular cone (CC) is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution around which the cone is invariant with respect to rotation [5, p. 102]. According to the definition, one can see that [31]

$$\mathscr{L}_\theta^n = \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1 \tan\theta \right\} \tag{1.1}$$

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is a CC in $\mathbb{R}^n$, where $\theta \in (0, \pi/2)$ is its half-aperture angle and $\|\cdot\|$ denotes the Euclidean norm. If $\theta = \pi/4$, then $\mathscr{L}_\theta^n$ reduces to the well-known second-order cone (SOC)

$$\mathscr{K}^n = \left\{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1\right\}.$$

When $\theta = \pi/4$ and $n = 1$, let $\mathscr{L}_\theta^n$ be the set of nonnegative real. In general, we can consider the following Cartesian product $\mathscr{L}_\theta$ of circular cons $\mathscr{L}_\theta^{n_i}$ in $\mathbb{R}^n$:

$$\mathscr{L}_\theta = \mathscr{L}_\theta^{n_1} \times \mathscr{L}_\theta^{n_2} \times \cdots \times \mathscr{L}_\theta^{n_r},$$

where $\mathscr{L}_\theta^{n_i}$ is defined as in (1.1) and $n_1 + n_2 + \cdots + n_r = n$ with $r, n_1, n_2, \cdots, n_r \ge 1$. In addition, let us also recall — cf. [6, p. 4], that for any nonempty set $\mathscr{C} \subseteq \mathbb{R}^n$, its dual cone is defined by

$$\mathscr{C}^* := \{y \in \mathbb{R}^n \mid \langle y, x\rangle \ge 0 \quad \text{for all} \quad x \in \mathscr{C}\}.$$

A cone $\mathscr{C}$ with $\mathscr{C} = \mathscr{C}^*$ is called a self-dual or symmetric cone. Note that unlike the SOC $\mathscr{K}^n$, the CC $\mathscr{L}_\theta^n$ is not a self-dual cone whenever $\theta \in (0, \pi/2)\backslash\{\pi/4\}$. Indeed, according to [31], we have

$$\left(\mathscr{L}_\theta^n\right)^* = \left\{y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|y_2\| \le y_1 \cot\theta\right\}.$$

Furthermore,

$$\mathscr{L}_\theta^* = \left(\mathscr{L}_\theta^{n_1}\right)^* \times \left(\mathscr{L}_\theta^{n_2}\right)^* \times \cdots \times \left(\mathscr{L}_\theta^{n_r}\right)^*.$$

With any CC we can associate generalized absolute value equations (CCGAVE) — viz.

$$Ax - B|x| = b, \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $x = (x_1^\intercal, \cdots, x_r^\intercal)^\intercal \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$, and $|x|$ denotes the absolute value of x corresponding to $\mathscr{L}_\theta$. To the best of our knowledge, CCGAVE (1.2) is first introduced in [19] and it reduces to the generalized absolute value equations associated with SOC (SOCGAVE) [9] when $\theta = \pi/4$. More specifically, if $\theta = \pi/4$ and $r = n$, CCGAVE (1.2) boils down to the generalized absolute value equations (GAVE) in $\mathbb{R}^n$ [23]. The GAVE is relevant to many scientific problems, such as linear complementarity problems, interval linear equations, quadratic programs and others — cf. [15, 17, 21, 22] and references therein. Any SOCGAVE is equivalent to a linear complementarity problem associated with SOC, which has numerous applications in control, finance and robust optimization [1, 4]. We are mainly interested in CCGAVE (1.2) because it is not only present the extensions of the GAVE and SOCGAVE but also gives an equivalent reformulation of linear complementarity problems associated with CC (CCLCP). The later plays a crucial role in the optimization community [2, 13].

Over the past two decades, a lot of efforts have been made in developing numerical methods and analyzing theoretical properties of the GAVE [3, 8, 14–17, 21, 25–30] and SOCGAVE [9, 10, 18, 20]. However, the study on CCGAVE (1.2) is still in its infancy, and to our knowledge, there is no work except for the results in [19]. This motivates us to do the job here.