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Long-Time Asymptotics for the Combined Nonlinear Schrödinger and Gerdjikov-Ivanov Equation

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Abstract. In this paper, the combined nonlinear Schrödinger and Gerdjikov-Ivanov (NLS-GI) equation with the Schwartz initial data is investigated. It is shown that the solution of NLS-GI equation can be expressed in terms of the solution associated with a Riemann-Hilbert (RH) problem. The long-time asymptotics is further obtained via the Deift-Zhou nonlinear steepest descent method.

AMS subject classifications: 35Q15, 35C20, 37K15, 37K40

Key words: Combined nonlinear Schrödinger and Gerdjikov-Ivanov equation, Riemann-Hilbert problem, Deift-Zhou steepest descent method, long-time asymptotics.

1. Introduction

In 1974, Manakov [28] first studied the long-time behavior of nonlinear wave equations by using the inverse scattering transform (IST) method. After that, Zakharov and Manakov [42] employed IST method in order to investigate the large-time asymptotics of the solutions of the nonlinear Schrödinger (NLS) equation with decaying initial value. The method also works for long-time behavior of integrable systems such as KdV, Landau-Lifshitz and the reduced Maxwell-Bloch system [3, 14, 33]. In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously obtain the long-time asymptotic behavior of the solution for the mKdV equation by deforming contours to reduce the original RH problem to a model whose solution can be determined via parabolic cylinder functions [8]. Since then, this method has been widely used in the focusing nonlinear Schrödinger equation, KdV equation, Boussinesq equation, Camassa-Holm (CH) equation, Degasperis-Procesi (DP) equation, Fokas-Lenells equation, Sasa-Satsuma equation, short-pulse equation, and Toda lattice [4–6, 9, 10, 19, 23, 26, 35, 37–39].

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The NLS equation

$$iu_t + u_{xx} + |u|^2 u = 0 \tag{1.1}$$

has important applications in a wide variety of fields, including nonlinear optics, theory of deep water waves, plasma physics, quantum field theory, etc [1,30,32]. The Eq. (1.1) has been studied by the inverse scattering method, Bäcklund transformation, Darboux transformation [40], which revealed its infinite conservation laws, and presented N-soliton solutions, rational solutions, quasi-periodic solutions — cf. Refs. [1,11,27,29,30,32]. In addition, new progress on the nonlinear Schrödinger equation has been made such as long-time asymptotics [7], initial-boundary value problems on a half-line or a finite interval [15,16] and other related studies [17,36].

The Gerdjikov-Ivanov (GI) equation [18] has the form

$$iu_t + u_{xx} - iu^2\bar{u}_x + \frac{1}{2}u^3\bar{u}^2 = 0,$$

where u is a complex valued function and $\bar{u}$ denotes the complex conjugation of u. It is well known that GI equation is one of the three types of derivative nonlinear Schrödinger equations. Many aspects of this equation, including bilinearization [22], gauge transformation [24], Darboux transformation [12], bi-Hamiltonian structure [13], hierarchy structure [20], Wronskian solution [21] and decomposition of the GI equation [41] have been studied.

Our research mainly focuses on the combined nonlinear Schrödinger and Gerdjikov-Ivanov equation

$$u_t = iu_{xx} - 2i|u|^2u + u^2\bar{u}_x + \frac{i}{2}|u|^4 u \tag{1.2}$$

with the Schwartz decaying initial value

$$u(t=0,x) = u_0(x) \in \mathscr{S}(\mathbb{R}). \tag{1.3}$$

The NLS-GI equation integrates the nonlinearities of NLS and GI equations. It is important to highlight the fact that the former maintains the integrability. However, related work has been less carried out. In recent years, only the soliton solutions [31] and the initial-boundary value problem [25] of the Eq. (1.2) have been studied via Riemann-Hilbert approach. Noting that the long-time asymptotics of NLS-GI equation remains unknown, we apply Deift-Zhou nonlinear steepest descent method to fill the gap in this area of work.

The structure of this manuscript is as follows. In Section 2, starting from the Lax pair of the Eq. (1.2), we construct Jost solutions and spectral matrix, and their analyticity and symmetries are further analyzed. Using the results of Section 2, we establish a RH problem on the jump contour $\Sigma^{(1)}$ in Section 3. In Section 4, we use the decomposition of the jump matrix, rational approximation of scattering data, and scaling transformation in order to change the RH problem into a model RH problem which can be solved via the Weber equation. In Section 5, based on the reconstruction formula between solution of the Eq. (1.2) and the RH problem, we derive the long-time asymptotics required. Section 6 contains a remark about the future work.