East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-275.150124

Vol. **15**, No. 1, pp. 163-184
February 2025

Long-Time Asymptotics for the Combined Nonlinear Schrödinger and Gerdjikov-Ivanov Equation

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Received 29 October 2023; Accepted (in revised version) 15 January 2024.

Abstract. In this paper, the combined nonlinear Schrödinger and Gerdjikov-Ivanov (NLS-GI) equation with the Schwartz initial data is investigated. It is shown that the solution of NLS-GI equation can be expressed in terms of the solution associated with a Riemann-Hilbert (RH) problem. The long-time asymptotics is further obtained via the Deift-Zhou nonlinear steepest descent method.

AMS subject classifications: 35Q15, 35C20, 37K15, 37K40

Key words: Combined nonlinear Schrödinger and Gerdjikov-Ivanov equation, Riemann-Hilbert problem, Deift-Zhou steepest descent method, long-time asymptotics.

1. Introduction

In 1974, Manakov [28] first studied the long-time behavior of nonlinear wave equations by using the inverse scattering transform (IST) method. After that, Zakharov and Manakov [42] employed IST method in order to investigate the large-time asymptotics of the solutions of the nonlinear Schrödinger (NLS) equation with decaying initial value. The method also works for long-time behavior of integrable systems such as KdV, Landau-Lifshitz and the reduced Maxwell-Bloch system [3, 14, 33]. In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously obtain the long-time asymptotic behavior of the solution for the mKdV equation by deforming contours to reduce the original RH problem to a model whose solution can be determined via parabolic cylinder functions [8]. Since then, this method has been widely used in the focusing nonlinear Schrödinger equation, KdV equation, Boussinesq equation, Camassa-Holm (CH) equation, Degasperis-Procesi (DP) equation, Fokas-Lenells equation, Sasa-Satsuma equation, short-pulse equation, and Toda lattice [4–6, 9, 10, 19, 23, 26, 35, 37–39].

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The NLS equation

$$iu_t + u_{xx} + |u|^2 u = 0 \tag{1.1}$$

has important applications in a wide variety of fields, including nonlinear optics, theory of deep water waves, plasma physics, quantum field theory, etc [1, 30, 32]. The Eq. (1.1) has been studied by the inverse scattering method, Bäcklund transformation, Darboux transformation [40], which revealed its infinite conservation laws, and presented N-soliton solutions, rational solutions, quasi-periodic solutions — cf. Refs. [1, 11, 27, 29, 30, 32]. In addition, new progress on the nonlinear Schrödinger equation has been made such as long-time asymptotics [7], initial-boundary value problems on a half-line or a finite interval [15, 16] and other related studies [17, 36].

The Gerdjikov-Ivanov (GI) equation [18] has the form

$$iu_t + u_{xx} - iu^2\bar{u}_x + \frac{1}{2}u^3\bar{u}^2 = 0,$$

where u is a complex valued function and $\bar{u}$ denotes the complex conjugation of u. It is well known that GI equation is one of the three types of derivative nonlinear Schrödinger equations. Many aspects of this equation, including bilinearization [22], gauge transformation [24], Darboux transformation [12], bi-Hamiltonian structure [13], hierarchy structure [20], Wronskian solution [21] and decomposition of the GI equation [41] have been studied.

Our research mainly focuses on the combined nonlinear Schrödinger and Gerdjikov-Ivanov equation

$$u_t = iu_{xx} - 2i|u|^2u + u^2\bar{u}_x + \frac{i}{2}|u|^4u \tag{1.2}$$

with the Schwartz decaying initial value

$$u(t=0,x) = u_0(x) \in \mathscr{S}(\mathbb{R}). \tag{1.3}$$

The NLS-GI equation integrates the nonlinearities of NLS and GI equations. It is important to highlight the fact that the former maintains the integrability. However, related work has been less carried out. In recent years, only the soliton solutions [31] and the initial-boundary value problem [25] of the Eq. (1.2) have been studied via Riemann-Hilbert approach. Noting that the long-time asymptotics of NLS-GI equation remains unknown, we apply Deift-Zhou nonlinear steepest descent method to fill the gap in this area of work.

The structure of this manuscript is as follows. In Section 2, starting from the Lax pair of the Eq. (1.2), we construct Jost solutions and spectral matrix, and their analyticity and symmetries are further analyzed. Using the results of Section 2, we establish a RH problem on the jump contour $\Sigma^{(1)}$ in Section 3. In Section 4, we use the decomposition of the jump matrix, rational approximation of scattering data, and scaling transformation in order to change the RH problem into a model RH problem which can be solved via the Weber equation. In Section 5, based on the reconstruction formula between solution of the Eq. (1.2) and the RH problem, we derive the long-time asymptotics required. Section 6 contains a remark about the future work.

2. Spectral Analysis

In this section, we introduce direct and inverse scattering transforms, which are the foundation of the Riemann-Hilbert problem.

2.1. Lax pair

The combined nonlinear Schrödinger and Gerdjikov-Ivanov equation (1.2) admits the following Lax pair:

$$\begin{aligned} Y_x &= UY, \\ Y_t &= VY, \end{aligned} \tag{2.1}$$

where $Y = Y(x,t,\lambda)$ is a matrix function called an eigenfunction and

$$\begin{aligned} U &= \begin{pmatrix} i\left(\lambda + \dfrac{1}{2}u\bar{u}\right) & (1+\lambda)u \\ \bar{u} & -i\left(\lambda + \dfrac{1}{2}u\bar{u}\right) \end{pmatrix}, \\ V &= \begin{pmatrix} A & (1+\lambda)(-2u\lambda + iu_x) \\ -2\bar{u}\lambda - i\bar{u}_x & -A \end{pmatrix}. \end{aligned}$$

Here, λ is an eigenvalue and

$$A = -2i\lambda^2 - i\lambda u\bar{u} - \frac{1}{2}(u_x\bar{u} - \bar{u}_x) + \frac{i}{4}u^2\bar{u}^2.$$

In order to obtain a more symmetrical Lax matrix, we introduce a gauge transformation — viz.

$$\hat{Y} = TY, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 - ik \end{pmatrix}, \quad k \neq -i,$$

where $\lambda = k^2$, cf. [31]. Subsequently, the Eq. (2.1) takes the form

$$\hat{Y}_x = \hat{U}\hat{Y}, \quad \hat{Y}_t = \hat{V}\hat{Y}, \tag{2.2}$$

where

$$\begin{aligned} \hat{U} &= \begin{pmatrix} i\left(k^2 + \dfrac{1}{2}u\bar{u}\right) & (1+ik)u \\ (1-ik)\bar{u} & -i\left(k^2 + \dfrac{1}{2}u\bar{u}\right) \end{pmatrix}, \\ \hat{V} &= \begin{pmatrix} B & -2iuk^3 - 2uk^2 - u_xk + iu_x \\ 2i\bar{u}k^3 - 2\bar{u}k^2 - \bar{u}_xk - i\bar{u}_x & -B \end{pmatrix}, \end{aligned}$$

and

$$B = -2ik^4 - iu\bar{u}k^2 - \frac{1}{2}(u_x\bar{u} - u\bar{u}_x) + \frac{i}{4}u^2\bar{u}^2 - iu\bar{u}.$$

For the sake of normalization, we introduce a matrix function $J = J(x,t,\lambda)$ defined by

$$\hat{Y} = JE, \tag{2.3}$$

where $E = e^{it(2\lambda^2-(x/t)\lambda)\sigma_3}$ and denote the phase function by

$$\phi(\lambda) \triangleq i\left(2\lambda^2 - \frac{x}{t}\lambda\right).$$

Using the transformation (2.3), we write the Lax pair (2.2) as

$$\begin{aligned} J_x &= i\lambda[\sigma_3, J] + \tilde{U}J, \\ J_t &= -2i\lambda^2[\sigma_3, J] + \tilde{V}J. \end{aligned} \tag{2.4}$$

Here, $[\sigma_3, J] = \sigma_3 J - J\sigma_3$ is the commutator and

$$\begin{aligned} &\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ &\tilde{U} = \begin{pmatrix} \frac{i}{2}u\bar{u} & (1+ik)u \\ (1-ik)\bar{u} & -\frac{i}{2}u\bar{u} \end{pmatrix}, \\ &\tilde{V} = \begin{pmatrix} -iu\bar{u}k^2 - \frac{1}{2}(u_x\bar{u} - u\bar{u}_x) + \frac{i}{4}u^2\bar{u}^2 - iu\bar{u} & -2iuk^3 - 2uk^2 - u_x k + iu_x \\ 2i\bar{u}k^3 - 2\bar{u}k^2 - \bar{u}_x k - i\bar{u}_x & iu\bar{u}k^2 + \frac{1}{2}(u_x\bar{u} - \bar{u}_x) - \frac{i}{4}u^2\bar{u}^2 + iu\bar{u} \end{pmatrix}. \end{aligned}$$

2.2. Analyticity and symmetry

We introduce matrix Jost solutions

$$J_\pm = \left(J_\pm^1, J_\pm^2\right) \tag{2.5}$$

for the space part of the Lax pair (2.4) which admits the asymptotic conditions

$$J_\pm \to I, \quad x \to \pm\infty, \tag{2.6}$$

where $J_\pm^l$, $l = 1,2$ denotes the l-th column of $J_\pm$ and I is the 2×2 identity matrix. Using the large-x asymptotic condition (2.6), we can turn the space part of (2.4) into the Volterra integral equations

$$J_\pm = I + \int_{\pm\infty}^{x} e^{-i\lambda(y-x)\hat{\sigma}_3}\tilde{U}(y)J_\pm dy, \tag{2.7}$$

where $e^{\hat{\sigma}_3}$ acts on a 2×2 matrix X as $e^{\hat{\sigma}_3}X = e^{\sigma_3}Xe^{-\sigma_3}$.

Under the condition (1.3), we are able to prove the existence and uniqueness of the Jost solutions $J_\pm$ by performing the standard procedures on the Volterra integral equations (2.7). Moreover, we can also prove that J_+^1 and J_-^2 can be analytically extended into the upper half-plane $\mathbb{C}^+$, and J_+^2, J_-^1 into the lower half-plane $\mathbb{C}^-$.

Now, we investigate the properties of $J_\pm$. Since the potential matrix $\tilde{U}$ is zero-trace, it implies that $\det J_\pm$ are independent of the variable x and t. In particular, evaluating $\det J_+$ at $x=+\infty$ and $\det J_-$ at $x=-\infty$, we have

$$\det J_\pm = 1.$$

Because the Lax pair (2.2) is a first-order linear homogeneous equation system, the functions (2.5) are linearly dependent. Consequently, we can represent the linear relation as

$$J_- = J_+ e^{-t\phi(\lambda)\sigma_3} S(\lambda) e^{t\phi(\lambda)\sigma_3}, \tag{2.8}$$

where $S(\lambda)$ is preset to the form below

$$S(\lambda) \triangleq \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

and

$$\det S(\lambda) = 1.$$

Assembling the analyticity of Jost solutions and their inverse matrices, we note that the scattering relation the analyticity of s_{11} in $\mathbb{C}^+$ and the analyticity of s_{22} in $\mathbb{C}^-$.

Straightforward matrix calculation immediately lead to the following proposition.

Proposition 2.1. *The above constructed Jost solutions $J_\pm$ and the scattering matrix $S(\lambda)$ enjoy the following symmetry relations:*

$$\begin{aligned}
&\sigma_1 \overline{J_\pm\left(x,t,\overline{\lambda}\right)}\sigma_1 = J_\pm(x,t,\lambda), \\
&\sigma_3 J_\pm^\dagger\left(x,t,\overline{\lambda}\right)\sigma_3 = J_\pm^{-1}(x,t,\lambda), \\
&\sigma_1 \overline{S\left(\overline{\lambda}\right)}\sigma_1 = S(\lambda), \\
&\sigma_3 S^\dagger\left(\overline{\lambda}\right)\sigma_3 = S^{-1}(\lambda),
\end{aligned}$$

where $A^\dagger$ denotes the conjugate transposition of matrix A.

We consider an asymptotic expansion

$$J = J_0 + \frac{J_1}{\lambda} + \frac{J_2}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty, \tag{2.9}$$

where J_0, J_1, J_2 are independent of λ. Substituting (2.9) into (2.4) and comparing the coefficients at λ, we obtain that J_0 is the unit matrix. Combining (2.9) and (2.4) implies that the potential is reconstructed by

$$u(x,t) = 2i \lim_{\lambda\to\infty} (\lambda J)_{12}.$$

3. Initial RH Problem

The scattering relation (2.8) can be deformed to

$$\left(\frac{J_-^2}{s_{22}}, J_+^1\right) = \left(J_+^2, \frac{J_-^1}{s_{11}}\right) e^{t\phi(\lambda)\hat{\sigma}_3} \begin{pmatrix} \dfrac{1}{s_{11}s_{22}} & -\dfrac{s_{21}}{s_{11}} \\ \dfrac{s_{12}}{s_{22}} & 1 \end{pmatrix}.$$

Note that the above equation is similar to the jump condition of a RH problem. We set

$$V(x,t,\lambda) = e^{t\phi(\lambda)\hat{\sigma}_3} \begin{pmatrix} \dfrac{1}{s_{11}s_{22}} & -\dfrac{s_{21}}{s_{11}} \\ \dfrac{s_{12}}{s_{22}} & 1 \end{pmatrix}$$

and subsequently define

$$m(x,t,\lambda) = \begin{cases} \left(\dfrac{J_-^2}{s_{22}}, J_+^1\right), & \lambda \in \mathbb{C}^+, \\ \left(J_+^2, \dfrac{J_-^1}{s_{11}}\right), & \lambda \in \mathbb{C}^-. \end{cases}$$

It is easy to check $m(x,t,\lambda)$ satisfies the following RH problem.

Riemann-Hilbert Problem 0.

(1) $m(x,t,\lambda)$ is analytic in $\mathbb{C}\backslash\Sigma$.

(2) On Σ, the boundary functions $m_\pm(x,t,\lambda)$ satisfy the jump condition

$$m_+(x,t,\lambda) = m_-(x,t,\lambda)V(x,t,\lambda), \quad \lambda \in \Sigma,$$

where

$$V(x,t,\lambda) = e^{t\phi(\lambda)\hat{\sigma}_3} \begin{pmatrix} 1-|r(\lambda)|^2 & -r(\lambda) \\ \overline{r(\lambda)} & 1 \end{pmatrix}, \quad r(\lambda) \triangleq \frac{s_{21}}{s_{11}}.$$

(3) Asymptotic condition: $m(x,t,\lambda) \to I, \lambda \to \infty$.

Remark 3.1. In this paper, we consider the case where s_{11} has no zeros — i.e. there are no solitons. Under this assumption, one can easily derive

$$|r(\lambda)| < 1. \tag{3.1}$$

4. Deformation of RH Problem

4.1. Stationary point and steepest descent lines

The phase function

$$\phi(\lambda) = i\left(2\lambda^2 - \frac{x}{t}\lambda\right)$$

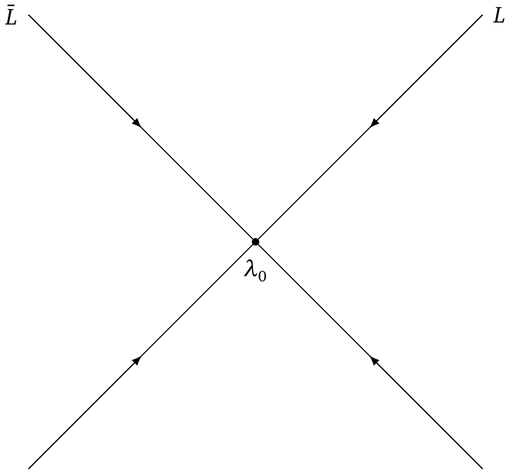

Figure 1: Steepest descent lines.

admits the first order point of stationary phase

$$\lambda_0 = \frac{x}{4t}.$$

Correspondingly, there are two steepest descent lines

$$\begin{aligned} L &= \left\{\lambda = \lambda_0 + \alpha e^{\pi/4i},\ \alpha \ge 0\right\} \cup \left\{\lambda = \lambda_0 + \alpha e^{5\pi/4i},\ \alpha \ge 0\right\}, \\ \bar{L} &= \left\{\lambda = \lambda_0 + \alpha e^{3\pi/4i},\ \alpha \ge 0\right\} \cup \left\{\lambda = \lambda_0 + \alpha e^{7\pi/4i},\ \alpha \ge 0\right\} \end{aligned}$$

as shown in Fig. 1.

4.2. Decomposition of jump matrix

We deform the RH problem with one jump matrix to the one with two matrices on $\Sigma^{(1)}$ by triangulating the jump matrix up and down where the jump contour $\Sigma^{(1)}$ is given by Fig. 2.

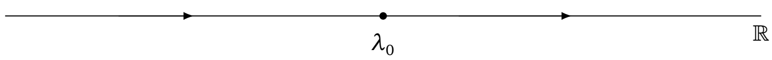

Figure 2: The oriented contour $\Sigma^{(1)}$.

Indeed, the jump matrix V enjoys the following triangular factorizations:

$$V = \begin{cases} \begin{pmatrix} 1 & -e^{2t\phi} r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2t\phi}\bar{r} & 1 \end{pmatrix}, & \lambda < \lambda_0, \\ e^{t\phi\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \dfrac{\bar{r}}{1-r^2} & 1 \end{pmatrix} \begin{pmatrix} 1-r^2 & 0 \\ 0 & \dfrac{1}{1-r^2} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{-r}{1-r^2} \\ 0 & 1 \end{pmatrix}, & \lambda > \lambda_0. \end{cases} \tag{4.1}$$

The importance of these factorizations is that they provide an algebraic separation of the oscillatory exponential factors $e^{\pm 2t\phi}$. Firstly, we need to deal with the central diagonal factor in the factorization (4.1) to be used for $\lambda > \lambda_0$. We introduce a scalar RH problem as follows.

Scalar Riemann-Hilbert Problem.

(1) $\delta(\lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(1)}$.

(2) Jump conditions

$$\begin{aligned} &\delta_+(\lambda) = \delta_-(\lambda), && \lambda < \lambda_0, \\ &\delta_+(\lambda) = \delta_-(\lambda)(1-r^2), && \lambda > \lambda_0. \end{aligned}$$

(3) Asymptotic condition: $\delta(\lambda) \to 1, \lambda \to \infty$.

This RH problem has been carefully studied — cf. [34]. Consider the complex scalar function defined by the formula

$$\delta(\lambda) = e^{(1/2\pi i)\int_{-\infty}^{\lambda_0} \ln(1-|r|^2)/(\xi-\lambda)d\xi}.$$

According to the Plemelj formula, this function provides the unique solution the scalar RH problem. Therefore, the diagonal matrix $\delta(\lambda)^{\sigma_3}$ is typically used in steepest descent theory to deal with the diagonal factor in (4.1). Note that direct calculations lead the following proposition.

Proposition 4.1. *$\delta(\lambda)$ is uniformly bounded in complex plane with*

$$\left(1-\|r\|_{L^\infty}^2\right)^{1/2} \leqslant |\delta(\lambda)| \leqslant \left(1-\|r\|_{L^\infty}^2\right)^{-1/2}.$$

Therefore, we can apply $\delta(\lambda)$ to modify the solution of RH Problem 0. More exactly, let us define the transformation

$$m^{(1)} = m\delta(\lambda)^{-\sigma_3}.$$

Then $m^{(1)}$ satisfies a new RH problem.

Riemann-Hilbert Problem 1.

(1) $m^{(1)}(x,t,\lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(1)}$.

(2) $m^{(1)}(x,t,\lambda)$ satisfies the jump condition

$$m_+^{(1)} = m_-^{(1)} V^{(1)}(\lambda), \quad \lambda \in \Sigma^{(1)},$$

where the jump matrix $V^{(1)}(\lambda)$ is defined by

$$V^{(1)} = \begin{cases} e^{t\phi\hat{\sigma}_3}\begin{pmatrix} 1 & -\delta_+^2 r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\bar{r} & 1 \end{pmatrix}, & \lambda < \lambda_0, \\ e^{t\phi\hat{\sigma}_3}\left(\begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\dfrac{\bar{r}}{1-r^2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \delta_+^2\dfrac{-r}{1-r^2} \\ 0 & 1 \end{pmatrix}\right), & \lambda > \lambda_0. \end{cases}$$

(3) Asymptotic condition: $m^{(1)}(x,t,\lambda) \to I, \lambda \to \infty$.

4.3. Rational approximation of scattering data

The coefficients

$$r(\lambda), \quad \bar{r}(\lambda), \quad \frac{r(\lambda)}{1-|r(\lambda)|^2}, \quad \frac{\bar{r}(\lambda)}{1-|r(\lambda)|^2},$$

which appear in the jump matrix $V^{(1)}$ do not have analytic continuations. To overcome this technical difficulty, we approximate the coefficients by rational functions

$$[r(\lambda)], \quad [\bar{r}(\lambda)], \quad \left[\frac{r(\lambda)}{1-|r(\lambda)|^2}\right], \quad \left[\frac{\bar{r}(\lambda)}{1-|r(\lambda)|^2}\right].$$

The following estimates are obtained similar to [8].

Proposition 4.2. *Let*

$$\rho(\lambda) = \begin{cases} r(\lambda), & \lambda < \lambda_0, \\ \dfrac{-r(\lambda)}{1-|r(\lambda)|^2}, & \lambda > \lambda_0. \end{cases}$$

We decompose $\rho(\lambda)$ into three parts — viz.

$$\rho = h_I + h_{II} + [\rho(\lambda)],$$

where the piecewise rational functions $[\rho(\lambda)]$ and h_{II} have analytic continuations to the steepest descent lines L and $\bar{L}$, but h_I does not. Moreover, we have

$$\begin{aligned} \|e^{-2t\phi}\delta^{-2}[\rho]\|_{L^1\cap L^2\cap L^\infty} &\leqslant ce^{-\epsilon t}, \quad \lambda \in L, \\ \|e^{-2t\phi}\delta^{-2}h_{II}\|_{L^1\cap L^2\cap L^\infty} &\leqslant ct^{-1}, \quad \lambda \in L, \\ \|e^{-2t\phi}\delta^{-2}h_{I}\|_{L^1\cap L^2\cap L^\infty} &\leqslant ct^{-1}, \quad \lambda \in \mathbb{R}, \end{aligned}$$

where ϵ is a positive number.

4.4. Analytic extension of jump matrix

The purpose of this section is to deform the RH problem on $\Sigma^{(1)} = \mathbb{R}$ to the one on $\Sigma^{(2)} = \mathbb{R} \cup L \cup \bar{L}$, see Fig. 3. We extend the jump matrix to the steepest descent lines L and $\bar{L}$. Thus we decompose the jump matrix as

$$\begin{aligned} V^{(1)} &= \begin{cases} \begin{pmatrix} 1 & \delta_+^2 \rho e^{2t\phi} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\bar{\rho} e^{-2t\phi} & 1 \end{pmatrix}, & \lambda < \lambda_0, \\ \begin{pmatrix} 1 & \delta_+^2 \rho e^{2t\phi} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\bar{\rho} e^{-2t\phi} & 1 \end{pmatrix}, & \lambda > \lambda_0 \end{cases} \\ &=: b_-^{-1} b_+, \end{aligned}$$

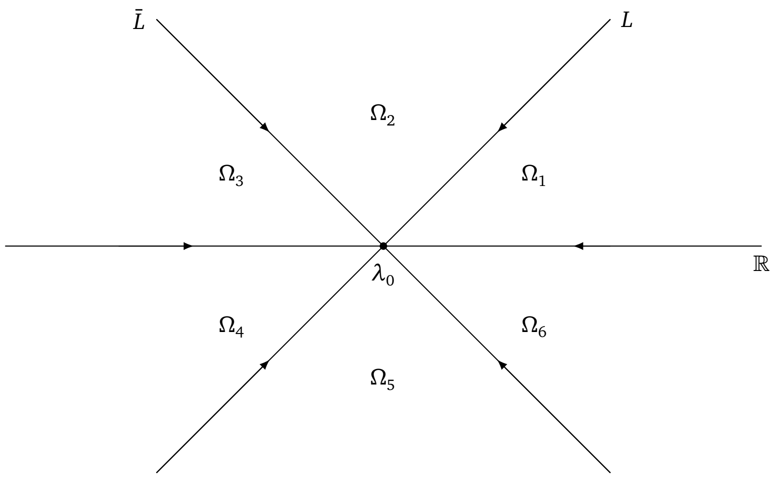

Figure 3: The oriented contour $\Sigma^{(2)}$.

where $b_\pm$ can be further represented in the form

$$\begin{aligned}
b_+ &= I + \omega_+ \\
&= \begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\bar{\rho}e^{-2t\phi} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \delta_-^{-2}h_{I,\bar{\rho}}e^{-2t\phi} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\left(h_{II,\bar{\rho}} + [\bar{\rho}]\right)e^{-2t\phi} & 1 \end{pmatrix} \\
&= b_+^0 b_+^a, \\
b_- &= I - \omega_- \\
&= \begin{pmatrix} 1 & \delta_+^2 \rho e^{2t\phi} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \delta_+^2 h_{I,\rho} e^{2t\phi} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta_+^2\left(h_{II,\rho} + [\rho]\right)e^{2t\phi} \\ 0 & 1 \end{pmatrix} \\
&= b_-^0 b_-^a.
\end{aligned}$$

Consequently, $V^{(1)}$ takes the form

$$V^{(1)} = b_-^{-1} b_+ = \left(b_-^a\right)^{-1} \left(b_-^0\right)^{-1} b_+^0 b_+^a.$$

In the above formulas, $(b_-^a)^{-1}$ and b_+^a can be analytically extended to L and $\bar{L}$, respectively. $(b_-^0)^{-1} b_+^0$ does not have an analytic continuation but decays fast in time. Therefore, we consider the following transformation:

$$m^{(2)} = m^{(1)}\psi,$$

where

$$\psi = \begin{cases} I, & \lambda \in \Omega_2 \cap \Omega_5, \\ \left(b_-^a\right)^{-1}, & \lambda \in \Omega_1 \cap \Omega_4, \\ \left(b_+^a\right)^{-1}, & \lambda \in \Omega_3 \cap \Omega_6, \end{cases}$$

and the corresponding regions are shown in Fig. 3.

The RH problem on $\Sigma^{(2)}$ is given by

Riemann-Hilbert Problem 2.

(1) $m^{(2)}$ is analytic in $\mathbb{C} \setminus \Sigma^{(2)}$.

(2) $m^{(2)}$ satisfies the jump condition

$$m_+^{(2)} = m_-^{(2)} V^{(2)}, \quad \lambda \in \Sigma^{(2)},$$

where the jump matrix is

$$V^{(2)}(x,t,\lambda) = \begin{cases} \left(b_-^0\right)^{-1} b_+^0, & \lambda \in \mathbb{R}, \\ \left(b_-^a\right)^{-1}, & \lambda \in L, \\ b_+^a, & \lambda \in \bar{L}. \end{cases}$$

(3) Asymptotic condition: $m^{(2)}(x,t,\lambda) \to I, \lambda \to \infty$.

Consequently, the solution of the combined nonlinear Schrödinger and Gerdjikov-Ivanov equation takes the form

$$u(x,t) = 2i \lim_{\lambda \to \infty} \left(\lambda m^{(2)}\right)_{12}. \tag{4.2}$$

Changing $\Sigma^{(2)}$ into

$$\Sigma^{(3)} \triangleq L \cap \bar{L} = \Sigma^{(2)} \setminus \mathbb{R},$$

we use it in evaluating the contribution of $h_I(\lambda)$ and $h_{II}(\lambda)$ to the solution of the RH problem. Taking into account results in [2], we represent the jump matrix in the form

$$V^{(2)} = \left(b_-^{(2)}\right)^{-1} b_+^{(2)}$$

and define

$$w_\pm^{(2)} = \pm\left(b_\pm^{(2)} - I\right),$$
$$w^{(2)} = w_+^{(2)} + w_-^{(2)}.$$

The Cauchy operators $C_\pm$ are defined as

$$(C_\pm f)(\lambda) = \lim_{\substack{\lambda' \to \lambda \\ \lambda' \in \pm \text{ side of } \Sigma^{(2)}}} \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{f(\xi)}{\xi - \lambda'} d\xi,$$

and

$$C_{w^{(2)}}f = C_+\left(f w_-^{(2)}\right) + C_-\left(f w_+^{(2)}\right).$$

According to Beals-Coifman theorem [2], we have

$$\begin{aligned}\mu^{(2)} &\triangleq (id - C_{w^{(2)}})^{-1} I \\ &= I + (id - C_{w^{(2)}})^{-1} C_{w^{(2)}} I,\end{aligned} \tag{4.3}$$

and the solution of the RH problem yields

$$m^{(2)}(\lambda) = I + \frac{1}{2\pi i}\int_{\Sigma^{(2)}} \frac{\mu^{(2)}(\xi) w^{(2)}(\xi)}{\xi - \lambda} d\xi. \tag{4.4}$$

Taking into account (4.2)-(4.4), we write

$$\begin{aligned}u(x,t) &= 2i \lim_{\lambda\to\infty}\left(\lambda m^{(2)}\right)_{12} \\ &= -\frac{1}{\pi}\left(\int_{\Sigma^{(2)}} \mu^{(2)}(\xi) w^{(2)}(\xi) d\xi\right)_{12} \\ &= -\frac{1}{\pi}\left(\int_{\Sigma^{(2)}} \left[(id - C_{w^{(2)}})^{-1} I\right] w^{(2)}(\xi) d\xi\right)_{12}.\end{aligned}$$

Using the method similar to the one exploited in rational approximation of the scattering data, we can show that the contributions to RH problem are mainly coming from the rational part of the scattering data. Therefore, the contribution of $h_I(\lambda)$ and $h_{II}(\lambda)$ is an infinitely small quantity of t. Consequently,

$$\begin{aligned}u(x,t) &= 2i\left(m_1^{(3)}\right)_{12} + \mathcal{O}(t^{-1}) \\ &= 2i \lim_{\lambda\to\infty}\left(\lambda m^{(3)}\right)_{12} + \mathcal{O}(t^{-1}),\end{aligned}$$

where

$$m^{(3)} = I + \frac{m_1^{(3)}}{\lambda} + \cdots$$

is the solution of following RH problem.

Riemann-Hilbert Problem 3.

(1) $m^{(3)}$ is analytic in $\mathbb{C} \setminus \Sigma^{(3)}$.

(2) The boundary value of $m^{(3)}$ satisfies the jump condition

$$m_+^{(3)} = m_1^{(3)} V^{(3)}, \quad \lambda \in \Sigma^{(3)}.$$

(3) Asymptotic condition: $m^{(3)} \to I, \lambda \to \infty$.

The jump matrix is shown in Fig. 4.

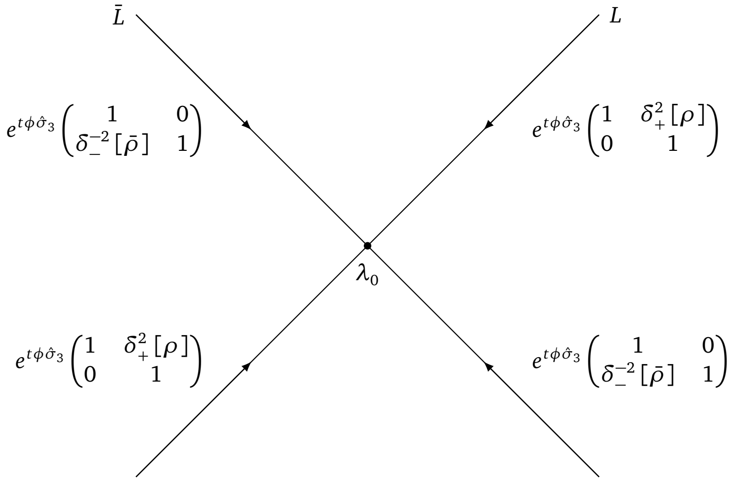

Figure 4: Oriented contour $\Sigma^{(3)}$ and jump matrix $V^{(3)}$.

4.5. Scaling transformation

First of all, we transform $\Sigma^{(3)}$ into the contour $\Sigma^{(4)}$, shown in Fig. 5, and take the scaling transformation

$$\lambda = (8t)^{-1/2}\tilde{\lambda} + \lambda_0.$$

Next, we define a scaling operator $N : L^p(\Sigma^{(3)}) \to L^p(\Sigma^{(4)})$ by

$$f(\lambda) \mapsto Nf(\lambda) = f\left((8t)^{-1/2}\tilde{\lambda} + \lambda_0\right).$$

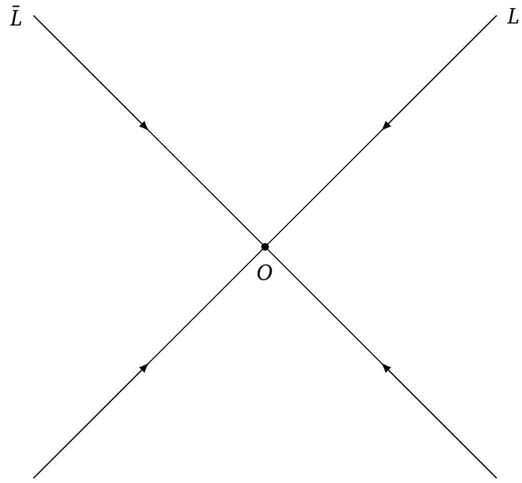

Figure 5: The oriented contour $\Sigma^{(4)}$.

The RH problem 3 yields

Riemann-Hilbert Problem 4.

(1) $m^{(4)}(x,t,\lambda)$ is analytic in $\mathbb{C}\setminus\Sigma^{(4)}$.

(2) The boundary value of $m^{(4)}$ satisfies the jump condition

$$m_+^{(4)} = m_1^{(4)} V^{(4)}, \quad \lambda\in\Sigma^{(4)}.$$

(3) Asymptotic condition: $m^{(4)}\to I, \lambda\to\infty$,

where

$$\begin{aligned} m^{(4)} &= Nm^{(3)} = m^{(3)}\left((8t)^{-1/2}\lambda+\lambda_0\right), \\ V^{(4)} &= NV^{(3)} = V^{(3)}\left((8t)^{-1/2}\lambda+\lambda_0\right). \end{aligned}$$

The solution of NLS-GI equation can be represented in the form

$$u(x,t) = \frac{i}{\sqrt{2t}}\left(m_1^{(4)}\right)_{12} + \mathcal{O}(t^{-1}),$$

where $m_1^{(4)}$ is defined by

$$\begin{aligned} m^{(4)} &= m^{(3)}\left((8t)^{-1/2}\lambda+\lambda_0\right) \\ &= I + \frac{m_1^{(3)}}{(8t)^{-1/2}\lambda+\lambda_0} + \cdots \\ &= I + \frac{m_1^{(4)}}{\lambda} + \cdots. \end{aligned}$$

For the sake of convenience, we isolate oscillations in the jump matrix $V^{(3)}$, viz.

$$\begin{aligned} V^{(4)} &= NV^{(3)} \\ &= N\left(\delta^{\hat{\sigma}_3} e^{t\phi\hat{\sigma}_3} V_0^{(3)}\right) \\ &= \left[N\left(\delta e^{t\phi}\right)\right]^{\hat{\sigma}_3} N V_0^{(3)}. \end{aligned}$$

By $\delta^0\delta^1$, we mean the result of the N operation on $\delta e^{t\phi}$, where

$$\begin{aligned} &\delta^0 = (8t)^{-i\nu/2} e^{\chi(\lambda_0)} e^{-2it\lambda_0}, \\ &\delta^1 = \lambda^{i\nu} e^{\chi(\lambda/\sqrt{8t}+\lambda_0)-\chi(\lambda_0)} e^{i\lambda^2/4}, \\ &\nu = \nu(\lambda_0) = -\frac{1}{2\pi}\log\left(1-|r(\lambda_0)|^2\right) > 0, \\ &\chi(\lambda) = -\frac{1}{2\pi i}\int_{-\infty}^{\lambda_0} \log|\lambda-\xi|\, d\log\left(1-|r(\xi)|^2\right). \end{aligned}$$

Removing the oscillating factor from the RH problem, we denote a new matrix function by

$$m^{(N)} = (\delta^0)^{-\hat{\sigma}_3} N^{(4)}.$$

A new jump matrix $V^{(N)}$ can be determined by the following jump condition:

$$m_+^{(N)} = \left(\delta^0\right)^{-\sigma_3} m_-^{(4)} \left(\delta^0\right)^{\sigma_3} \left(\delta^0\right)^{-\sigma_3} V^{(4)} \left(\delta^0\right)^{\sigma_3} = m_-^{(N)} V^{(N)}.$$

This gives

$$u(x,t) = \frac{i}{\sqrt{2t}} \left(\delta^0\right)^2 \left(m_1^{(N)}\right)_{12} + \mathcal{O}\left(t^{-1}\right),$$

where $m^{(N)}$ is the solution of the following RH problem.

Riemann-Hilbert Problem 5.

(1) $m^{(N)}(x,t,\lambda)$ is analytic in $\mathbb{C} \setminus \Sigma^{(N)}$.

(2) The boundary value of $m^{(N)}$ at $\Sigma^{(N)} \triangleq \Sigma^{(4)}$ satisfies the jump condition

$$m_+^{(N)} = m_1^{(N)} V^{(N)}, \quad \lambda \in \Sigma^{(N)}.$$

(3) Asymptotic condition: $m^{(N)} \to I, \lambda \to \infty$.

Note that $V^{(N)}$ is shown in Fig. 6.

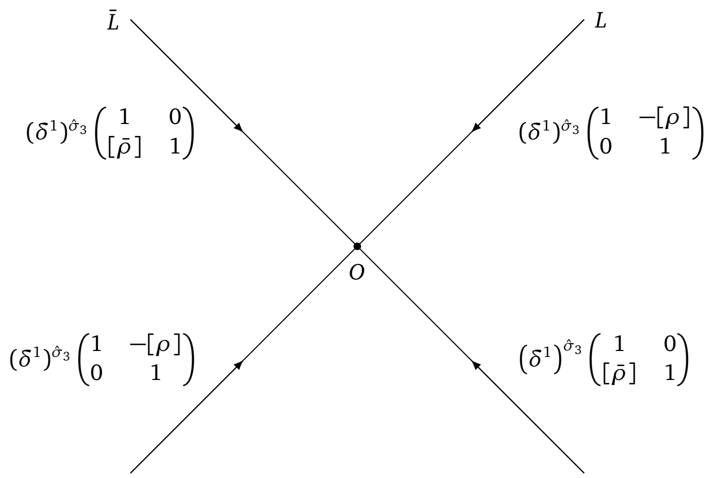

Figure 6: Oriented contour $\Sigma^{(N)}$ and jump matrix $V^{(N)}$.

Now, we transform the jump curve $\Sigma^{(N)}$ into Σ^{∞} by taking the limit to attribute RH problem on the phase point λ_0. If $t \to \infty$, then

$$\begin{aligned}
&\frac{\lambda}{\sqrt{8t}} + \lambda_0 \to \lambda_0, \\
&\delta^1 \to \lambda^{i\nu} e^{i\lambda^2/4}, \\
&V^{(N)}(x,t,\lambda) \to V^{\infty}(x,t,\lambda), \\
&[\rho]\left(\frac{\lambda}{\sqrt{8t}} + \lambda_0\right) \to \rho(\lambda_0) \triangleq [\rho(\lambda_0)],
\end{aligned}$$

where $V^{\infty}(x,t,\lambda)$ is shown in Fig. 7.

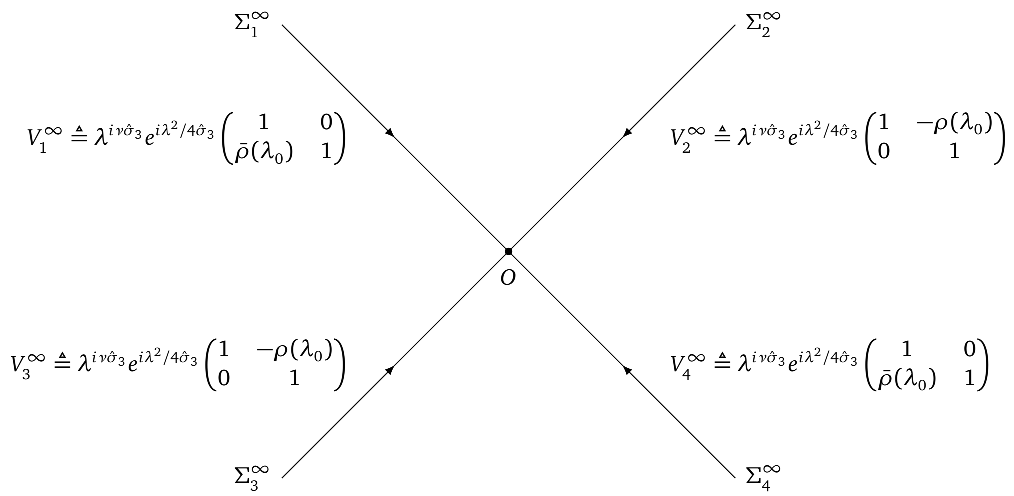

Figure 7: Oriented contour Σ^{∞} and jump matrix V^{∞}.

Moreover, we can prove that

$$\|V^{(N)} - V^{\infty}\|_{L^1 \cap L^{\infty}} \leqslant ct^{-1/2} \log t,$$

and obtain the RH problem of $(V^{\infty}, \Sigma^{\infty})$.

Riemann-Hilbert Problem 6.

(1) $m^{\infty}(x,t,\lambda)$ is analytic in $\mathbb{C} \setminus \Sigma^{\infty}$.

(2) The boundary value of m^{∞} at $\Sigma^{\infty} \triangleq \Sigma^{(N)}$ satisfies the jump condition

$$m_+^{\infty} = m_1^{\infty} V^{\infty}, \quad \lambda \in \Sigma^{\infty}.$$

(3) Asymptotic condition: $m^{\infty} \to I, \lambda \to \infty$.

Subsequently, the solution of the original equation is represented as

$$u(x,t) = \frac{i}{\sqrt{2t}} \left(\delta^0\right)^2 \left(m_1^\infty\right)_{12} + \mathcal{O}\left(t^{-1}\log t\right), \tag{4.5}$$

where m_1^∞ is defined by

$$m^\infty = I + \frac{m_1^\infty}{\lambda} + \cdots .$$

We set $\Sigma^e \triangleq \Sigma^\infty \cup \mathbb{R}$ with modified contour direction and

$$V^e = \begin{cases} V^\infty, & \lambda \in \Sigma_1^e \cup \Sigma_3^e, \\ (V^\infty)^{-1}, & \lambda \in \Sigma_3^e \cup \Sigma_4^e, \\ I, & \lambda \in \Sigma_5^e \cup \Sigma_6^e. \end{cases}$$

Now we divide the complex plain by Σ^e into six domains $\hat{\Omega}_j$, cf. Fig. 8.

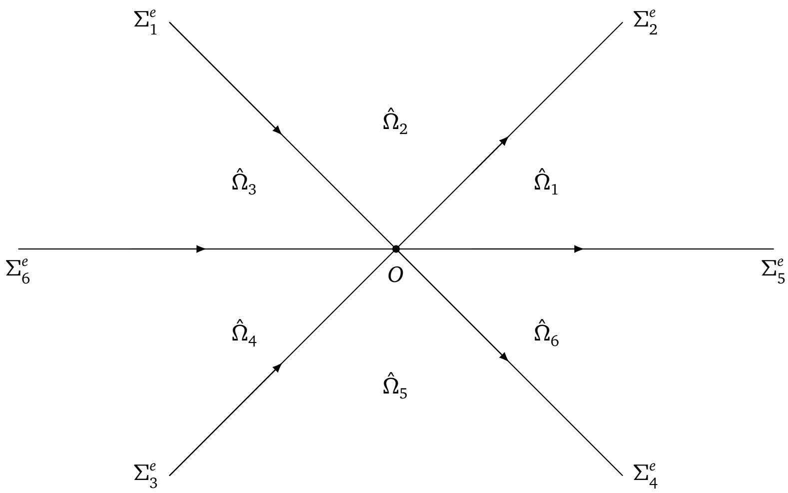

Figure 8: The oriented contour Σ^e and the domains.

It is clear that m^∞ satisfies the RH problem (Σ^e, V^e). Defining the function γ by

$$\gamma(\lambda) = \begin{cases} \lambda^{-i\nu\sigma_3}, & \lambda \in \hat{\Omega}_2 \cup \hat{\Omega}_5, \\ \lambda^{-i\nu\sigma_3}(V^e)^{-1}, & \lambda \in \hat{\Omega}_1 \cup \hat{\Omega}_3, \\ \lambda^{-i\nu\sigma_3} V^e, & \lambda \in \hat{\Omega}_4 \cup \hat{\Omega}_6 \end{cases}$$

and considering the transformation

$$\hat{m} = m^\infty \gamma^{-1},$$

we obtain the following result:

$$\begin{aligned}&\hat{m}_+ = \hat{m}_-\hat{V}, \qquad \lambda \in \mathbb{R},\\ &\hat{m}\lambda^{-i\nu\sigma_3} \to I, \quad \lambda \to \infty,\end{aligned}$$

where $\hat{V} = \gamma_- V^\infty \gamma_+^{-1}$. In other words, $\hat{V}$ takes the form

$$\hat{V} = \begin{cases} I, & \lambda \in \Sigma_1^e \cup \Sigma_2^e \cup \Sigma_3^e \cup \Sigma_4^e, \\ e^{i\lambda^2/4\hat{\sigma}_3} V(\lambda_0), & \lambda \in \Sigma_5^e \cup \Sigma_6^e = \mathbb{R}. \end{cases}$$

Under the transformation $\Phi = \hat{m}e^{(i\lambda^2/4)\sigma_3}$, a standard RH problem can be defined as

Riemann-Hilbert Problem 7.

(1) Φ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

(2) $\Phi_+ = \Phi_- V(\lambda_0)$, $\lambda \in \mathbb{R}$.

(3) $\Phi e^{-(i\lambda^2)/4\sigma_3} \lambda^{-i\nu\sigma_3} \to I, \lambda \to \infty$.

This kind of RH problems can be solved by using the Weber equation and parabolic-cylinder function.

5. Long-Time Asymptotics

In terms of the jump condition of the last RH problem and the Liouville theorem, we can derive that $(d\Phi/d\lambda + (1/2)i\lambda\sigma_3\Phi)\Phi^{-1}$ is a constant matrix with respect to λ. Hence, there exist constants β_{ij}, $i,j = 1,2$ such that

$$\left(\frac{d\Phi}{d\lambda} + \frac{1}{2}i\lambda\sigma_3\Phi\right)\Phi^{-1} = \frac{1}{2}i\left[\sigma_3, m_1^\infty\right] \triangleq \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}. \tag{5.1}$$

Comparing two sides of (5.1) yields

$$\left(m_1^\infty\right)_{12} = -i\beta_{12}. \tag{5.2}$$

Substituting (5.2) into (4.5) gives

$$u(x,t) = \frac{1}{\sqrt{2t}}\left(\delta^0\right)^2 \beta_{12} + \mathscr{O}\left(t^{-1}\log t\right). \tag{5.3}$$

Parabolic-cylinder functions can be used for calculating β_{12}. Let

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

Starting with (5.1), we note that if $\operatorname{Im}\lambda > 0$, then

$$\begin{aligned}
\Phi_{11}^{+} &= e^{-3\pi\nu/4} D_a\left(\lambda e^{-3\pi i/4}\right),\\
\Phi_{12}^{+} &= e^{3\pi\nu/4}\beta_{21}^{-1}\left[\partial_\lambda D_{-a}\left(\lambda e^{-\pi i/4}\right) - \frac{i\lambda}{2} D_{-a}\left(\lambda e^{-\pi i/4}\right)\right],\\
\Phi_{21}^{+} &= e^{-3\pi\nu/4}\beta_{12}^{-1}\left[\partial_\lambda D_a\left(\lambda e^{-3\pi i/4}\right) + \frac{i\lambda}{2} D_a\left(\lambda e^{-3\pi i/4}\right)\right],\\
\Phi_{22}^{+} &= e^{\pi\nu/4} D_{-a}\left(\lambda e^{-3\pi i/4}\right),
\end{aligned} \tag{5.4}$$

and if $\operatorname{Im}\lambda > 0$, then

$$\begin{aligned}
\Phi_{11}^{-} &= e^{\pi\nu/4} D_a\left(\lambda e^{\pi i/4}\right),\\
\Phi_{12}^{-} &= e^{-3\pi\nu/4}\beta_{21}^{-1}\left[\partial_\lambda D_{-a}\left(\lambda e^{3\pi i/4}\right) - \frac{i\lambda}{2} D_{-a}\left(\lambda e^{3\pi i/4}\right)\right],\\
\Phi_{21}^{-} &= e^{\pi\nu/4}\beta_{12}^{-1}\left[\partial_\lambda D_a\left(\lambda e^{\pi i/4}\right) + \frac{i\lambda}{2} D_a\left(\lambda e^{\pi i/4}\right)\right],\\
\Phi_{22}^{-} &= e^{-3\pi\nu/4} D_a\left(\lambda e^{3\pi i/4}\right),
\end{aligned} \tag{5.5}$$

where $a = -i\nu(\lambda_0)$ and $D_a(\zeta) = D_a(e^{-(3/4)\pi i}\lambda)$ is the standard parabolic cylinder function satisfying the Weber equation

$$\partial_\zeta^2 D_a(\zeta) + \left(\frac{1}{2} - \frac{\zeta^2}{4} + \beta_{12}\beta_{21}\right) D_a(\zeta) = 0.$$

We rewrite the jump condition in the following form:

$$\Phi_-^{-1}\Phi_+ = V(\lambda_0) = \begin{pmatrix} 1 - |r(\lambda_0)|^2 & -r(\lambda_0) \\ \overline{r(\lambda_0)} & 1 \end{pmatrix}.$$

Assembling (5.4)-(5.5) and the above formula, we arrive at the relation

$$\overline{r(\lambda_0)} = \Phi_{11}^{-}\Phi_{21}^{+} - \Phi_{21}^{-}\Phi_{11}^{+} = \frac{(2\pi)^{1/2} e^{\pi i/4} e^{-\pi\nu/2}}{\beta_{12}\Gamma(-a)},$$

where $\Gamma(a)$ is Gamma function. Thus

$$\beta_{12} = \frac{(2\pi)^{1/2} e^{\pi i/4} e^{-\pi\nu/2}}{\overline{r(\lambda_0)}\Gamma(-a)}. \tag{5.6}$$

Substituting (5.6) into (5.3), we immediately extract the following asymptotics.

Theorem 5.1. *As $t \to \infty$, such that $|\lambda_0| = |x/4t| < M$,*

$$u(x,t) = \frac{\tau(\lambda_0)}{\sqrt{t}} e^{-ix^2/4t - i\nu(\lambda_0)\log 8t} + \mathcal{O}\left(t^{-1}\log t\right),$$

where

$$\tau(\lambda_0)=\frac{\pi^{1/2}e^{\pi i/4}e^{-\pi\nu/2}e^{2\chi(\lambda_0)}}{\overline{r(\lambda_0)}\Gamma(-a)}.$$

Its modulus and angle are

$$\begin{aligned}|\tau(\lambda_0)|^2&=-\frac{1}{4\pi}\ln\big(1-|r(\lambda_0)|^2\big),\\ \arg\tau(\lambda_0)&=\frac{\pi}{4}+\arg\Gamma\big(i\nu(\lambda_0)\big)-\arg r(\lambda_0)\\ &\quad+\frac{1}{\pi}\int_{-\infty}^{\lambda_0}\ln|\lambda-\lambda_0|d\ln\big(1-|r(\lambda)|^2\big),\end{aligned}$$

respectively.

Combining the above results leads to the main theorem.

Theorem 5.2. *Suppose that the initial value u_0 belongs to the Schwartz space, $\lambda_0=x/4t$ is a stationary phase point, and $|\lambda_0|<M$ with a fixed constant $M>1$. Then the long-time asymptotics for the NLS-GI equation* (1.2) *is given by*

$$u(x,t)=\frac{\tau(\lambda_0)}{\sqrt{t}}e^{-ix^2/4t-i\nu(\lambda_0)\log 8t}+\mathcal{O}\left(t^{-1}\log t\right). \tag{5.7}$$

6. Conclusion

The crucial technique for obtaining the asymptotics (5.7) is the classical Deift-Zhou steepest decent method. This method has been widely used in the study of the long-time behavior of numerous differential equations. We see that the result is determined not by the equation itself but the spectral problem corresponding to the equation. This inspires us to consider a new concept, which we tentatively call the asymptotic invariant. The purpose of this concept is to investigate the relationship between different spectral problems and their associated asymptotic behavior. As a corollary, the NLS equation and the GI equation give an example of producing integrable system by combination. In terms of the asymptotic invariant, we may be able to skip the procedures and get to the conclusion about the produced equation directly.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 12171098, 12261053, 11571079, 11701322), and by the Natural Science Foundation of Shanghai (Grant No. 14ZR1403500).

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