

On the Parameterized Polynomial Inverse Eigenvalue Problem

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Abstract. The paper focuses on the solvability and computability of the parameterized polynomial inverse eigenvalue problem (PPIEP). Employing multiparameter eigenvalue problems, we establish a sufficient solvability condition for the PPIEP. Three numerical methods are used to solve PPIEPs. The first one is the Newton method based on locally smooth QR-decomposition with the column pivoting and the second the Newton method based on the smallest singular value. In order to reduce the computational cost of computing the smallest singular values and the corresponding unit left and right singular vectors in each iteration, we approximate these values by using one-step inverse iterations. Subsequently, we introduce another method — viz. a Newton-like method based on the smallest singular value. Each of three methods exhibits locally quadratic convergence under appropriate conditions. Numerical examples demonstrate the effectiveness of the methods proposed.

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1. Introduction

Over the years, there has been significant discourse surrounding inverse eigenvalue problems, primarily driven by their wide-ranging applications. The applications encompass various fields, including but not limited to dynamic behavior control of damped mass-spring systems [22], structural design [27], finite element model updating [10, 11], and circuit theory [34]. Corresponding to a wide range of applications, there exist diverse categories of inverse eigenvalue problems — e.g. parameterized, structured, partially described, and others [9]. In this work, we study a particular inverse eigenvalue problem — viz. the parameterized polynomial inverse eigenvalue problem (PPIEP), which is determined as follows.

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Problem 1.1 (PPIEP). Let $A_m \in \mathbf{R}^{n \times n}$ be a non-singular matrix and $\{A_j^{(q)}\}_{j=0}^{mn} \in \mathbf{R}^{n \times n}$, $q = 0, 1, \dots, m-1$. For any distinct complex numbers $\lambda_1, \lambda_2, \dots, \lambda_{mn}$, find a vector $c = (c_1, c_2, \dots, c_{mn})^T \in \mathbf{R}^{mn}$ or $\mathbf{C}^{mn}$ such that the polynomial eigenvalue problem

$$[\lambda^m A_m + \lambda^{m-1} A_{m-1}(c) + \dots + \lambda A_1(c) + A_0(c)]x = 0$$

with the matrices

$$A_q(c) = A_0^{(q)} + \sum_{j=1}^{mn} c_j A_j^{(q)}, \quad q = 0, 1, \dots, m-1 \quad (1.1)$$

has the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{mn}$.

The following notations will be used throughout the discussion. The sets of all real and complex $m \times n$ matrices are respectively denoted by $\mathbf{R}^{m \times n}$ and $\mathbf{C}^{m \times n}$ and we write $\mathbf{R}^m$ for $\mathbf{R}^{m \times 1}$ and $\mathbf{C}^m$ for $\mathbf{C}^{m \times 1}$. Besides, $\mathcal{U}_n$ and $\mathcal{R}_n$ refer to the sets of all $n \times n$ unitary and all $n \times n$ upper-triangular matrices, respectively. In addition, I is the identity matrix of a suitable size, e_j the j -th column vector of I , and $\|\cdot\|$ represents the Euclidean vector norm or the induced matrix norm. Moreover, if A is an $m \times n$ matrix, then A^T and $\sigma_{\min}(A)$ respectively denote the transpose and the smallest singular value of A .

The PPIEP is a class of problems. It contains the parameterized standard inverse eigenvalue problem (PSIEP) — i.e. if $m = 1$, $A_m = I$, the parameterized generalized inverse eigenvalue problem (PGIEP) — i.e. if $m = 1$, $A_m = A_1(c)$, and the parameterized quadratic inverse eigenvalue problem (PQIEP) — i.e. if $m = 2$. The theory and applications of these problems have been extensively investigated in the past decades. The reader can consult [6, 7, 9, 24, 41, 42] for PSIEP studies, [1, 12, 15, 16, 18–20, 27, 31] for PGIEP studies, and [21–23, 38, 39] for PQIEP studies. However, to the best of our knowledge, there is a lack of pertinent information regarding high-order PPIEPs. Similar to all inverse eigenvalue problems, there are two fundamental questions concerning the PPIEP associated problems — viz. their solvability and computability.

The solvability analysis of PSIEP, PGIEP and PQIEP is believed to be beneficial in analyzing the solvability of the PPIEP, due to the unique relationship that exists among these four problems. Additionally, the numerical methods for solving PSIEP, PGIEP and PQIEP can contribute to the development of numerical methods for solving the PPIEP. Hence, we take PQIEP as an illustrative example to analyze the solvability of PSIEP, PGIEP and PQIEP as well as the existing numerical techniques employed for solving these problems.

For the solvability issue, Xiang and Dai [38] presented a sufficient condition for the existence of a solution of PQIEP by using the theory of multiparameter eigenvalue problem [3]. Using the same theory, Ji [26] and Dai *et al.* [15] proposed sufficient conditions for the existence of solutions of PSIEP and PGIEP earlier.

For the computability issue, most of the numerical algorithms with locally quadratic convergence (under appropriate conditions) are frequently constructed by formulating an equivalent nonlinear system and then employing Newton method to solve it. Based on the determinant evaluations proposed by Lancaster [29] and Biegler-König [6] and further