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A Difference Mixed Finite Element Method for the Three Dimensional Poisson Equation

Qi Zhang, Pengzhan Huang* and Yinnian He

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, P.R. China.

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Abstract. A difference mixed finite element method based on the finite element pair $((P_0\times P_1),(P_0\times P_1),(P_1\times P_0))\times(P_1\times P_1)$ for the three dimensional Poisson equation is studied. The method combines a mixed finite element discretization based on $(P_0,P_0,P_1)\times P_1$-element in (x,y)-plane and a finite difference discretization based on $(P_1,P_1,P_0)\times P_1$-element in z-direction. This allows to transmit the finite element solution of the three dimensional Poisson equation in the direction (x,y,z) into a series of finite element solution of the two dimensional Poisson equation in the direction (x,y). Moreover, error estimates for the discrete approximation are derived. Numerical tests show the accuracy and efficiency for the method proposed.

AMS subject classifications: 65N12, 65N22

Key words: Poisson equation, difference mixed finite element, error estimate, inf-sup condition, finite element pair.

1. Introduction

Finite element and finite difference methods are efficient ways to solve partial differential equations — cf. Refs. [2–4, 10, 16, 31, 34]. It is known that these methods made great contributions to numerical solutions of partial differential equations, and each approach has its own merits. In contrast to finite difference methods, finite element methods are more suitable to deal with regions with complex geometry. On the other hand, finite difference methods are easier to implement.

Despite a considerable increase in available computing power in recent decades, still there are difficulties in solving three dimensional (3D) partial differential equations, such as overcoming the discretization of 3D space, constructing basis functions, etc. To reduce the computational complexity of solving 3D equations, He *et al.* [15] constructed a difference finite element (DFE) based on the $P_1\times P_1$-conforming finite element discretization in the (x,y)-plane and finite difference discretization in the z-direction. Notably, they found

*Corresponding author. *Email addresses:* `zqmath0901@163.com` (Q. Zhang), `hpzh@xju.edu.cn` (P. Huang), `heyn@mail.xjtu.edu.cn` (Y. He)

that the assembly of the stiffness matrix by such a DFE method is easier than the finite element method. In fact, this method can be used in many real application problems. In particular, for 3D primitive equations of the ocean [17, 18, 28], it benefits from the use of a discretization in the z-direction in the way. We note that the horizontal velocity and the vertical velocity are components of velocity in primitive equations.

Later on, the DFE method was subsequently used by Feng *et al.* [12] to solve the 3D heat equation. Recently, the authors of the present work, proposed a DFE method for 3D steady Stokes equations based on finite element pair $((P_1^b, P_1^b, P_1) \times P_1) \times (P_1 \times P_0)$ [13]. It was also proven that this finite element pair satisfies the LBB condition in 3D domain. Furthermore, Feng *et al.* [14] used the DFE method with the same finite element pair to solve 3D steady Navier-Stokes equations. In addition, Lu *et al.* [26, 27] presented a stabilized DFE method for 3D steady Stokes equations and Navier-Stokes equations with help of the finite element pair $((P_1, P_1, P_1) \times P_1) \times (P_1 \times P_0)$, and shown that this finite element pair satisfies a weak LBB condition in 3D domain.

In many practical situations, it is important to compute dual variables of partial differential equations more accurately. For example, the gradient of the solution is the dual variable in case of the Poisson equation, whereas the stress or pressure variable is the dual variable in case of elasticity equation. In these situations, it is natural to introduce a mixed finite element method. It is worth noting that mixed finite element spaces must be chosen carefully so that they satisfy the LBB condition for the mixed method to be stable. Nowadays, there are many special mixed finite element spaces, including Nédélec [29], Raviart and Thomas [8, 11, 30], Brezzi *et al.* [5–7], Chen and Douglas [9]. These mixed spaces do not contain the lowest order element for the $((P_0, P_0), P_1)$ pair. Moreover, for the two dimensional Poisson equation, Shi *et al.* [33] have considered the finite element pair $((P_0, P_0), P_1)$ which satisfied the LBB condition. Further, this finite element pair has undergone some evolution and has been further developed [1, 22, 23, 25, 35–39]. Besides, based on this finite element pair, Hou *et al.* [19–21] considered the superconvergence and two-grid methods. Shi and Yang [32] obtained unconditionally optimal error estimates of a new mixed finite element method for nonlinear Schrödinger equations.

Inspired by the works of He *et al.* [15] and Shi *et al.* [33], here we construct and analyze a difference mixed finite element (DMFE) method for the 3D Poisson equation with the finite element pair $((P_0 \times P_1), (P_0 \times P_1), (P_1 \times P_0)) \times (P_1 \times P_1)$. The calculation domain in the method proposed is a cuboid, and the side surface has to be perpendicular to the horizontal surface. Moreover, we prove that this finite element pair complies with the LBB condition. Compared to the method in [12], the DMFE method performs finite difference discretization in the z-direction and finite element discretization in the (x, y)-plane on the basis of introducing intermediate flux which has to be only quadratic integrable. Besides, the lower order finite element is applied in the DMFE method.

The rest of this paper is organized as follows. In Section 2, we introduce necessary notations and recall the classical and other mixed variational formulations for the 3D Poisson equation. Section 3 describes the finite difference method based on the finite element pair $(P_1, P_1, P_0) \times P_1$ for the z-direction discretization. In Section 4, the DMFE solution pair $(\mathbf{u}_h, p_h)$ based on $((P_0 \times P_1), (P_0 \times P_1), (P_1 \times P_0)) \times (P_1 \times P_1)$-element of the 3D Poisson