East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2023-268.120324

Vol. **15**, No. 2, pp. 314-343
May 2025

Postprocessing-Based A Posteriori Error Estimation for Spectral Galerkin Approximations of Fourth-Order Boundary Value Problems

Zhe Li[1], Tao Sun[2] and Lijun Yi[1,*]

[1]*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.*
[2]*School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China.*

Received 21 October 2023; Accepted (in revised version) 12 March 2024.

Abstract. A postprocessing-based a posteriori error estimates for the spectral Galerkin approximation of one-dimensional fourth-order boundary value problems are developed. Our approach begins by introducing a novel postprocessing technique aimed at enhancing the accuracy of the spectral Galerkin approximation. We prove that this postprocessing step improves the convergence rate in both L^2- and H^2-norm. Using postprocessed superconvergence results, we construct several a posteriori error estimators and prove that they are asymptotically exact as the polynomial degree increases. We further extend the postprocessing technique and error estimators to more general one-dimensional even-order equations and to multidimensional fourth-order equations. The results of numerical experiments illustrate the efficiency of the error estimators.

AMS subject classifications: 65N35, 65L10, 41A10

Key words: Spectral Galerkin method, fourth-order boundary value problem, superconvergent postprocessing, a posteriori error estimation.

1. Introduction

Many significant physical phenomena, such as elasticity [25], fluid mechanics [18] and thin film growth [23], can be effectively modeled by fourth-order boundary value problems (BVPs). To solve such BVPs, there have been developed a variety of numerical approaches, including finite difference and finite element methods — cf. [12,32,36] and [11,13,22,42], respectively.

Spectral methods are an important tools for solving differential and integral equations — cf. [4, 15, 16, 24, 29, 31, 37–39] and references therein. They demonstrate exceptional accuracy and, in recent years, they have been widely used in solving fourth-order BVPs

*Corresponding author. *Email address:* `ylj5152@shnu.edu.cn` (L.J. Yi)

[2, 3, 10, 17, 27, 28, 30, 33, 34, 41, 43]. It is well-known that a posteriori error estimates play a pivotal role in the development of adaptive algorithms with minimal computational overhead. While the theory of a posteriori error estimation for finite element methods in the context of fourth-order BVPs has been extensively explored and documented [1,5,6,14, 26,35], there appears to be comparatively less emphasis on a posteriori error estimation for spectral methods for fourth-order problems. We only note the work [8], which is focused on a posteriori error estimation for the Legendre spectral Galerkin method applied to one-dimensional bi-harmonic equations. The a posteriori error estimators proposed there, solely depend on the right-hand side function of the equation and have been proven to provide two-sided bounds for actual errors.

In this paper we will consider the spectral Galerkin method for the fourth-order BVP of the form

$$\begin{aligned} &\partial_x^4 u - \partial_x(\alpha \partial_x u) + \beta u = f, \quad x \in I := (-1,1), \\ &u(\pm 1) = \partial_x u(\pm 1) = 0, \end{aligned} \tag{1.1}$$

where $\alpha, \beta \geq 0$. For analysis, we assume that the functions α, β and f are sufficiently smooth, so that the exact solution u of (1.1) is also a smooth function.

The main objective of this paper is to introduce and analyze asymptotically exact a posteriori error estimators for the spectral Galerkin approximation of the problem (1.1). In the context of spectral methods, asymptotic exactness implies that the ratio between the error estimator and the true error approaches 1 as the polynomial degree tends to infinity. It is worth noting that a posteriori error estimation based on postprocessing for spectral Galerkin method of second-order BVPs in one dimension has been studied in [40], and the error estimators were shown to be asymptotically exact. More recently, the approach introduced in [40] has been extended in [19–21] to address a posteriori error estimation for the spectral Galerkin methods applied to fractional differential equations.

In this paper, we extend the postprocessing technique introduced in [40] to the fourth-order problem (1.1). Roughly speaking, the key idea of the postprocessing technique is to add several higher order polynomials of degree greater than N to the already obtained spectral Galerkin approximation u_N of degree N, cf. the Eq. (3.10). The main contributions of this article can be summarized as follows:

1. For the one-dimensional fourth-order equation (1.1), we prove that the proposed postprocessing technique can improve the convergence rates for both L^2- and H^2-errors in spectral Galerkin approximations.

2. We introduce several a posteriori error estimators based on the differences between the standard Galerkin and postprocessed Galerkin approximations of the one-dimensional fourth-order equation. We demonstrate the asymptotic exactness of these error estimators as the polynomial degree N increases.

3. We extend the postprocessing technique and a posteriori error estimators, originally designed for one-dimensional fourth-order equations, to handle more general one-dimensional even-order equations and multidimensional fourth-order equations.