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Difference Finite Element Methods Based on Different Discretization Elements for the Four-Dimensional Poisson Equation

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Abstract. This paper proposes difference finite element (DFE) methods for the Poisson equation in a four-dimensional (4D) region $\omega \times (0, L_4)$. The method converts the Poisson equation in a 4D region into a series of three-dimensional (3D) subproblems by the finite difference discretization in $(0, L_4)$ and deals with the 3D subproblems by the finite element discretization in ω. In performing the finite element discretization, we select different discretization elements in the region ω: hexahedral, pentahedral, and tetrahedral elements. Moreover, we prove the stability of the DFE solution u_h and deduce the first-order convergence of u_h with respect to the exact solution u under H^1-error. Finally, three numerical examples are given to verify the accuracy and effectiveness of the DFE method.

AMS subject classifications: 35J05, 65N06, 65N30

Key words: 4D Poisson equation, difference finite element method, hexahedral element, pentahedral element, tetrahedral element.

1. Introduction

The Poisson equation is an essential class of elliptic PDEs used to solve various practical problems, including the heat conduction and electric potential distribution. Numerical approaches commonly used for solving this equation include finite element (FE) and finite difference (FD) methods. The main principle of FE methods is to discretize the continuous problem in the variational form [1–8, 15, 18, 19], while the main principle of the FD method is to directly approximate the differential terms in the local partial differential equation,

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transforming the differential equation into algebraic equations [9, 14, 16, 24, 25, 27–30]. FD methods are usually simpler and easier to implement, whereas FE methods are more suitable for solving complex regional problems.

Since FE and FD methods rely on mesh, the solution of the Poisson equation in dimensions higher than three by these methods represents a significant challenge. In this paper, we propose a DFE method specifically designed for solving the 4D Poisson equation by combining FE and FD methods. The method discretizes the 4D Poisson equation into a series of 3D subproblems through FD discretization and then solves these 3D problems using the FE method. By examining the coefficient matrix formed by the DFE method, it can be observed that during the assembly of the coefficient matrix, we only need to compute the stiffness and mass matrices of the 3D Poisson equation once. Therefore, we transform the computation of the 4D problem into the calculation of the 3D Poisson problem, thereby reducing the computational complexity.

The DFE method was first proposed by He *et al.* [13] in order to solve the 3D Poisson equation. The method transforms the 3D Poisson problem into a series of 2D elliptic problems by the finite difference discretization in the z-direction. Then, the DFE method has been applied to the 3D heat, Stokes, Navier-Stokes, and MHD equations [10–12, 22, 23]. So, the validity of the DFE method in 3D regions has been fully verified. The aim of this paper is to explore the applicability, stability, and convergence of the DFE method for 4D problems. First, we consider the FD discretization in the x_4-direction to transform the 4D Poisson problem into a series of 3D subproblems and show the stability of the FD solution u_τ. Then, we choose different discretization elements corresponding to different basis functions to discretize the 3D subproblems — viz. $Q_1(x_1,x_2,x_3)$-element in hexahedral mesh, $P_1(x_1,x_2)\times P_1(x_3)$-element in pentahedral mesh, and $P_1(x_1,x_2,x_3)$-element in tetrahedral mesh. Furthermore, we show the first-order convergence of the DFE solutions u_h to the exact solution u in the H^1-norm. Numerical examples confirm the effectiveness of the DFE method for the 4D Poisson equation.

The structure of this paper is as follows. Section 2 reviews the FE method for the 3D Poisson equation based on three different discretization elements and recalls the definition and properties of a projection operator $R_h: X\to X_h$. Section 3 considers the discretization of the 4D Poisson equation in $(0,L_4)$ by the FD method. Then, we define the FD solution

$$u_\tau=\sum_{m=1}^{l_4-1}u^m(x_1,x_2,x_3)\psi_m(x_4)$$

and prove the stability of u_τ and the error estimate for u_τ. In Section 4, we show the details of the DFE method of the 4D Poisson equation and define the DFE solution

$$u_h=\sum_{l=1}^{N}\sum_{m=1}^{l_4-1}u_l^m\phi_l(x_1,x_2,x_3)\psi_m(x_4).$$

The stability of the DFE solution u_h and H^1-error estimates for u_h are proved. Finally, three numerical examples of the 4D Poisson equation are presented to test the correctness of the theoretical results and the finiteness of the DFE method in Section 5.