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A High-Order Arbitrary Lagrangian-Eulerian Oscillation-Free Discontinuous Galerkin Scheme for One-Dimensional Compressible Multi-Material Flows

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Abstract. A high-order arbitrary Lagrangian-Eulerian (ALE) oscillation-free discontinuous Galerkin scheme for one-dimensional compressible multi-material flows is proposed. It couples a conservative equation in the volume-fraction model and the Euler equations of the fluid mixture dynamics. The mesh velocity is obtained by an adaptive mesh method, which automatically concentrates mesh nodes near the regions with large gradients. This greatly reduce numerical dissipation at material interfaces. Besides, the resolution of the solution at material interfaces can be also improved. To control the oscillations, we add damping terms into the weak formulation of the system. Neither parameter in new terms has to be artificially adjusted, and the difficulties with discontinuous solutions and complexities of designing limiters can be avoided. The scheme can be efficiently applied to compressible multi-material flows with essentially non-oscillatory property and its steps are more concise than in indirect ALE methods. Although here we only consider a third-order scheme, the method can be extended to any other order by choosing suitable basis functions. Examples demonstrate the third-order accuracy and good performance of the scheme for one-dimensional compressible multi-material flows.

AMS subject classifications: 65M60, 35Q31, 76M10
Key words: Compressible multi-material flow, arbitrary Lagrangian-Eulerian, oscillation-free DG scheme, adaptive mesh method, damping term.

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1. Introduction

The hydrodynamics of multi-material flows is of great interest in computational fluid dynamics (CFD) and in many other fields of national economy and energy. The simulation of compressible multi-material flows has a great theoretical and practical significance for understanding various phenomena in nuclear physics, biological engineering, and other areas. Therefore, numerical methods for compressible multi-material flows attracted substantial attention in recent years.

Generally speaking, there are two kinds of classical approaches for dealing with multi-material flows — viz. Eulerian and the Lagrangian methods. The Eulerian methods have strong robustness when solving the problems with large deformations [36, 41], and high-resolution schemes, like essentially non-oscillatory (ENO) schemes, perform well in the simulation of single-material flows [34,35]. On the other hand, in multi-material situations the Eulerian methods are not quite able to capture the clear physical interfaces because of numerical inaccuracies caused by the transport calculation. The Lagrangian methods can catch the material interfaces clearly in the simulation of multi-material flows, but in case of large deformations, the large mesh distortion may lead to errors or the interruptions of computational codes. For avoiding large mesh distortions in the simulations, the mesh rezoning phase and the variables remapping phase need to be introduced to the method. The Lagrangian methods have been studied by Despres and Mazeran [11], Maire *et al.* [28], Dobrev *et al.* [12]. Combining the advantage of the Eulerian method with the one of the Lagrangian method, Hirt *et al.* [18] developed an arbitrary Lagrangian-Eulerian (ALE) method in which the mesh can move with the arbitrary specified velocity. The ALE methods can flexibly simulate the flows with large deformations and moving regions. There are two kinds of ALE methods — viz. indirect and direct ones. The former consist of three steps — viz Lagrangian step, mesh rezoning step, and variables remapping step [5, 32], while the later has two steps — viz. the one for obtaining the mesh velocity and the other for discretizing the system [15, 44]. The direct ALE methods without remapping are concise and can be used to construct the high-order schemes more easily. In this work we construct and study a direct ALE method for multi-material flows.

Note that the implementation of conservative methods for multi-material flows may encounter certain difficulties — e.g. numerical oscillations arising at material interfaces. In recent decades, significant progress has been made in the development of numerical schemes for simulating multi-material flows — cf. [1–3, 20, 21]. In particular, Abgrall and Karni [2] have reviewed some numerical algorithms and pointed out common key features of the algorithms considered. Among these schemes, an extended system consisting of Euler and additional equations has been used to describe the evolutions of fluid parameters such as the volume-fraction, the level set function or the ratio of specific heats of the fluid mixture. To maintain the pressure equilibrium and eliminate spurious oscillations near the regions involving the material interfaces, various models such as the volume-fraction model have been proposed. Besides, they introduced a number of numerical schemes with an additional non-conservative equation which was used to describe fluid mixture — cf. [2]. In the present work, we construct a conservative scheme for multi-material flows combined