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# The Randomized Milstein Scheme for Stochastic Volterra Integral Equations with Weakly Singular Kernels

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**Abstract.** This paper focuses on the randomized Milstein scheme for approximating solutions to stochastic Volterra integral equations with weakly singular kernels and non-differentiable drift coefficients. An essential component of the error analysis involves the utilization of randomized quadrature rule for stochastic integral to avoid the Taylor expansion in the drift coefficient. Finally, we implement the simulation of multiple singular stochastic integral in the numerical experiment by applying the Riemann-Stieltjes integral.

**AMS subject classifications**: 65C30, 60H10

**Key words**: Stochastic Volterra integral equation, randomized Milstein scheme, strong convergence, randomized quadrature rule.

## 1. Introduction

Over the past several decades, stochastic Volterra integral equations (SVIEs) have been widely applied in various areas, including mathematical finance, physics, biology, and engineering [6, 20]. However, such equations are nonlinear and it is difficult to determine their explicit solutions, so that it is necessary to develop numerical schemes [5, 11, 17, 18, 27]. In particular, numerical methods for stochastic Volterra integral equations with weakly singular kernels (SVIEwWSKs) have been widely studied. In particular, a $\theta$-Euler scheme and a Milstein scheme for SVIEwWSKs are investigated in [16]. Besides, the Euler-Maruyama (EM) scheme and a Milstein scheme of a more general form of SVIEs were studied in [24]. Later on, a fast EM scheme for weakly singular SVIEs with variable exponent has been introduced in [15]. The EM scheme for weakly singular stochastic fractional integro-differential equations was investigated in [3], while the fast EM scheme for SVIEs with singular and

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Hölder continuous kernels was studied in [28]. For more information about stochastic differential equations (SDEs) the reader can consult [9,29–36]. In addition, randomized Euler and Runge-Kutta schemes for deterministic differential equations have been studied in [4,7,8,10,12,25,26]. A randomized EM scheme for scalar SDEs with Carathéodory type drift coefficient functions was investigated in [23]. Furthermore, randomized EM schemes for scalar SDEs with drift coefficients Lipschitz continuous with respect to the space variable but only measurable with respect to the time variable were introduced in [21,22], while a randomized Euler scheme of SDEs with drift and diffusion coefficients perturbed by a deterministic noise was studied in [19].

Inspired by the drift-randomized Milstein scheme in [13], we apply a randomized scheme to the classical Milstein scheme, which leads to a randomized Milstein scheme for SVIEwWSKs. Compared with the drift coefficient function in [16], the function $b$ here is not necessarily differentiable and this can be observed in numerical simulations. Note that the drift and diffusion coefficients in [13] are temporal Hölder continuous, hence it is more challenging to cope with the singular kernels in the drift and diffusion coefficients $(t-s)^{-\alpha}$ and $(t-s)^{-\beta}$, since in our work they tend to infinity as $s$ tends to $t$. The main result of this paper shows that, under Assumptions 2.1 and 2.2, the convergence rate of the randomized Milstein scheme for SVIEsWSKs does not exceed $\min\{1-2\beta, 1-\alpha\}$. Moreover, testing the convergence rate still remains a problem in [14,16] since the simulation of the multiple singular stochastic integrals

$$I=\int_{t_{j-1}}^{t_j}(t_n-s)^{-\beta}\sigma'\left(X_{j-1}^h\right)\left(\sum_{k=1}^{j-1}\int_{t_{k-1}}^{t_k}\left[(s-r)^{-\beta}-(t_{j-1}-r)^{-\beta}\right]\sigma\left(X_{k-1}^h\right)dB_r\right)dB_s$$

is a significant challenge. This problem is resolved by using the Riemann-Stieltjes integral. It will be discussed in more detail in Section 4.

The rest of the paper is structured as follows. In Section 2, we introduce notations, assumptions and a few important lemmas useful in the proofs. Section 3 aims to get the final convergence rate of the randomized Milstein scheme. In Section 4, we consider a numerical example to validate the effectiveness of the theoretical results.

## 2. Preliminaries

If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $x\in R^n$, then $|x|$ is its Euclidean norm. If $A$ is a matrix, let $|A|=\sqrt{\mathrm{trace}(A^TA)}$ be its trace norm. If $a,b$ are real numbers, then $a\vee b:=\max\{a,b\}$, $a\wedge b:=\min\{a,b\}$. Let $\mathbb{S}_n=\{1,\dots,n\}$, $\mathbb{S}_n^0=\{0,1,\dots,n\}$ for any $n\in\mathbb{N}$. If $s\in\mathbb{R}$, then $\lfloor s\rfloor$ refers to the integer part of $s$. Denote by $C^2(\mathbb{R}^d)$ the family of twice continuously differentiable functions in $\mathbb{R}^d$. For $p>0$ and $t\geq 0$, we denote by $L^p(\Omega,\mathscr{F},\mathbb{P})=L^p(\Omega)$ the family of $R^d$-valued random variables $X$ such that

$$\|X\|_{L^p(\Omega)}:=(\mathbb{E}|X|^p)^{1/p}=\left(\int_\Omega|X(w)|^p d\mathbb{P}(w)\right)^{1/p}<\infty.$$