East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2024-072.150824

Vol. **15**, No. 3, pp. 565-590
August 2025

Solving the Inverse Source Problem of the Fractional Poisson Equation by MC-fPINNs

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Received 1 May 2024; Accepted (in revised version) 15 August 2024.

Abstract. In this paper, we effectively solve the inverse source problem of the fractional Poisson equation using a Monte Carlo sampling-based PINN method (MC-fPINN). We construct two neural networks $u_{NN}(x;\theta)$ and $f_{NN}(x;\psi)$ to approximate the solution $u^*(x)$ and the forcing term $f^*(x)$ of the fractional Poisson equation. To optimize these networks, we use the Monte Carlo sampling method and define a new loss function combining the measurement data and underlying physical model. Meanwhile, we present a comprehensive error analysis for this method, along with a prior rule to select the appropriate parameters of neural networks. Numerical examples demonstrate the great accuracy and robustness of the method in solving high-dimensional problems up to 10D, with various fractional orders and noise levels of the measurement data ranging from 1% to 10%.

AMS subject classifications: 68T07, 65M12, 62G05
Key words: Fractional Poisson equation, MC-fPINN, error analysis, inverse source problem.

1. Introduction

The fractional partial differential equations (fPDEs) play a crucial role in the mathematical modeling of anomalous phenomena in science and engineering, including hydrology [16], viscoelasticity [12], and turbulent flow [3, 6]. In practical applications, we often need to recover some information missed in these fPDEs, including the diffusion coefficients, initial or boundary data, or source terms, especially for various problems in physics [4] and biology [11]. However, solving inverse problems for the fractional Poisson equation is a challenging task because of the non-locality and singularity of the fractional Laplacian operator.

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More recently, with the significant advancements in deep learning techniques applied to computational vision and natural language processing, neural network architectures have been used to solve inverse problems for fPDEs. For example, Pang *et al*. extended PINNs [14] to fPINNs [13] to solve the inverse problem of identifying parameters in the partial advection diffusion equation. In their method, the integer derivative is calculated using automatic differentiation while the fractional derivative is approximated by traditional numerical discretization, which leads to a high computational cost, especially for high-dimensional problems. Later on, Yan *et al*. [18] proposed a Bayesian inversion with neural operator (BINO) approach to solve the Bayesian inverse problem of the time fractional subdiffusion equation, of which the diffusion coefficient can be recovered. In addition, they presented a Laplace fPINNs method to identify the diffusion coefficient function in the time-fractional diffusion equation [19]. This method first transforms the original equation into a restricted problem in Laplace space and then solves it using PINNs. However, despite the successful application of BINO and Laplace-fPINNs to time-fractional problems, the methods have not been used in space-fractional and high-dimensional problems.

To deal with space-fractional derivative and higher-dimensional problems more efficiently, Guo *et al*. [7] proposed a Monte Carlo sampling-based PINN method to identify the parameters in the fractional advection-diffusion equation. Unlike to fPINNs, the MC-fPINN calculates fractional derivatives using Monte Carlo sampling instead of traditional methods. This reduces computational cost and enables their application to higher-dimensional problems. Furthermore, Feng *et al*. [5] introduced a MC-Nonlocal-PINNs approach, extending the original MC-fPINNs for more general nonlocal operators. In this paper, we extend this idea to address the inverse source problems for the fractional Poisson equation in both low and high dimensions. In summary, our main contributions are as follows:

1. Except for the neural network for approximating the solutions of the fractional Poisson equation, we represent the forcing term function as another fully-connected neural network. To optimize these two neural networks, we use the Monte Carlo sampling method mentioned in MC-fPINN and define a new loss function containing regularization terms for residuals of fPDEs and measurement data. We effectively address the inverse source problems related to the fractional Poisson equation, even in high-dimensional cases. In computational experiments, we show the high accuracy and reliability of our method by varying the fractional orders while introducing noise to the measurement data at levels between 1% and 10%.

2. We provide a systematical error analysis for the MC-fPINN method for this inverse problem and a guideline for choosing suitable neural network parameters, such as the total number of nonzero weights, depth, and samples. This guarantees that the error in estimation remains consistent and manageable in order to attain the desired level of accuracy.

The rest of this paper is organized as follows. In Section 2, we describe the inverse problem of the fractional Poisson equation with the Dirichlet boundary condition and the notation that will be used in this work. In Section 3, an inverse MC-fPINN method is