East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2024-056.270624

Vol. **15**, No. 3, pp. 591-614
August 2025

An HOC Finite Difference Scheme for the Steady Natural Convection Problem Based on the Velocity-Vorticity Method

Tao Wang[1,*], Tiegang Liu[2] and Kun Wang[1]

[1] *School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, P.R.China.*
[2] *LMIB and School of Mathematical Sciences, Beihang University, Beijing 100191, P.R.China.*

Received 5 April 2024; Accepted (in revised version) 27 June 2024.

Abstract. A high-order compact finite difference scheme for solving natural convection problems using velocity-vorticity formulation of the incompressible Navier-Stokes equations is presented. The basic idea of the method is to regard all controlling equations as the Poisson-type. We construct a fourth-order finite difference scheme for the velocity-vorticity equation based on the nine-point stencils for each Poisson-type equation. Next we give an example with an exact solution to verify that the scheme has the fourth-order accuracy. Finally, numerical solutions for the model problem of natural convection in a square heating cavity are presented to show the reliability and effectiveness of this method.

AMS subject classifications: 65Y04, 65Z05, 65N06, 65D25

Key words: Navier-Stokes equation, Boussinesq hypothesis, velocity-vorticity method, fourth-order compact scheme, natural convection problem.

1. Introduction

Natural convection problems exists in fluid flow and heat transfer. Industrial applications of natural convection have many aspects, such as solar collectors, building insulation, air cooling equipment, turbulent shear flow [20, 29, 30, 33, 36]. In general, one describes these systems using the incompressible Navier-Stokes (NS) equations. Since exact solutions of the corresponding equations are not available, researchers consider finding numerical solutions the problem mentioned.

The development of efficient and effective computational methods for natural convection problems has practical significance. Numerical methods for incompressible NS equations are usually divided into primitive variables [2, 17, 26, 27] and non-primitive variables methods. Primitive variables methods are widely used but their main deficiency is the

*Corresponding author. *Email address:* `wangtao@nmu.edu.cn` (T. Wang)

necessity to deal with the relationship between the pressure and velocity during the solution process. To overcome the problems caused by the pressure, two typical non-primitive variable approaches are usually used — viz. velocity-vorticity and streamfunction-vorticity methods. Note that there are a variety of works devoted to high-order compact difference schemes for streamfunction-vorticity methods — cf. Refs. [4, 5, 7, 11, 19, 28, 31, 32]. These methods have a high accuracy, high resolution, and high efficiency. Nevertheless, streamfunction-vorticity methods can not be extended to three-dimensional (3D) problems.

As an alternative, velocity-vorticity methods for incompressible NS equations has been extensively developed in the past 60 years. Fasel [10] used a second-order velocity-vorticity method to study the stability of fluid flows for two-dimensional (2D) boundary layers. These methods take vorticity and velocity as solving variables and do not concern with the pressure, thus reducing the coupling between unknown variables. The advantage of such methods is that they can be directly extended to three dimensional problems. Therefore, they have been broadly used to compute 2D and 3D steady and non-steady flows problems [1, 3, 6, 8, 12–14, 21–25, 34]. Orlandi [25] adopted a second-order velocity-vorticity method to solve the incompressible NS equations with staggered grid discretization. Guj and Stella [12] obtained a 2-nd order velocity-vorticity formula and studied the steady solution of 2D driven cavity flow with Reynolds number (Re) not beyond 5000. Dacles and Hafez [6] proposed a second-order velocity-vorticity method and used it to solve 3D driving cavity by a staggered grid. Daube [23] solved the velocity-vorticity incompressible NS equations to describe cylindrical cavities. Hester [14] gave a second-order velocity-vorticity method for solving 2D incompressible NS equations and provided the stability analysis of the scheme. Overall, we note that most of the above mentioned studies employ second order accuracy schemes. Few works on high-order compact schemes for solving 2D steady-state NS/Boussinesq equations with the velocity-vorticity method have been disclosed. In view of high accuracy, high resolution and high efficiency of high-order compact schemes for solving 2D steady-state NS/Boussinesq equations by velocity-vorticity methods, we can better capture small- and micro-scale flow structures in natural convection problems.

This work aims to develop a fully fourth-order compact difference scheme with the nine-point stencils to solve the 2D steady-state velocity-vorticity NS/Boussinesq equations governing natural convection problem in a square cavity. The outline of the paper is as follows. In Section 2, we propose a nine-point fourth-order compact scheme for solving 2D steady-state NS/Boussinesq equations by a velocity-vorticity method. In Section 3, the accuracy of the proposed scheme is verified by using an exact solution problem. We also simulate numerically the natural convection problem in a square cavity by using the scheme presented and give the benchmark solutions from Rayleigh number Ra 10^3 to 10^6. Section 4 contains main conclusions.

2. Numerical Method

The Navier-Stokes/Boussinesq equation representing the steady-state of a 2D incompressible fluid flow and heat transfer has the following velocity-vorticity form: