

Stable and Conservative Finite Difference Time-Domain Methods for Rotating Nonlinear Klein-Gordon Equation

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Abstract. We consider numerical discretizations for nonlinear Klein-Gordon/wave equations in a rotating frame. Due to the strong centrifugal forces in the model, non-proper spatial discretizations of the rotating terms (under finite difference or finite element) would lead to numerical instability that cannot be overcome by standard time averages. We identify a class of boundary-stable type finite difference discretizations. Based on it, we propose several stable and accurate finite difference time-domain schemes with discrete conservation laws. Extensive numerical experiments and simulations are done to understand the significance of the model and the proposed schemes.

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1. Introduction

The second order wave equation

$$\partial_{tt}u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + f(u(\mathbf{x}, t)) = 0$$

in general describes various wave phenomena in reality. If the background of the matters is rotating under a constant angular velocity $\gamma \neq 0$, to describe the physical system, it is often convenient to change the lab frame into a rotating frame that rotates with the background — i.e. $\mathbf{x} \rightarrow A(t)\mathbf{x}$ with the rotational matrix, e.g. for $\mathbf{x} \in \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} \cos(\gamma t) & -\sin(\gamma t) \\ \sin(\gamma t) & \cos(\gamma t) \end{pmatrix}.$$

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This will make the rotational wave system still be described by a Hamiltonian equation which is preferred physically and mathematically, but the price is to introduce two inertial forces into the wave model

$$\partial_{tt}u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + f(u(\mathbf{x}, t)) - 2\gamma(\mathbf{x} \times \nabla)\partial_t u(\mathbf{x}, t) + \gamma^2(\mathbf{x} \times \nabla)^2 u(\mathbf{x}, t) = 0.$$

In this paper, we shall investigate the numerical issues towards such rotating wave models. A precise example that we shall focus on is the Klein-Gordon model introduced in the following.

The Klein-Gordon equation is a classical model widely applied in quantum physics for describing dynamics of a high energy system. As one of the applications, it is particularly popular for modelling cosmic superfluid [17, 21, 23, 24, 28–30]. To describe a rotating galaxy, the rotating nonlinear Klein-Gordon (RKG) equation was first introduced in [23, 35]

$$\frac{1}{c^2}\partial_{tt}\psi(\mathbf{x}, t) - \Delta\psi(\mathbf{x}, t) + \left(\frac{mc}{\hbar}\right)^2\psi(\mathbf{x}, t) + m\lambda|\psi(\mathbf{x}, t)|^2\psi(\mathbf{x}, t) - R_{Co} - R_{ce} = 0, \quad (1.1)$$

where $\psi = \psi(\mathbf{x}, t) : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{C}$ is the unknown scalar field for $d = 2, 3$ with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ or $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, and

$$R_{Co} = i\frac{2\gamma}{c^2}L_z\partial_t\psi(\mathbf{x}, t), \quad R_{ce} = \frac{\gamma^2}{c^2}L_z^2\psi(\mathbf{x}, t)$$

with L_z the z-component of the angular momentum operator

$$L_z = -i\hbar(x\partial_y - y\partial_x).$$

Here the first order angular momentum operator term R_{Co} describes the Coriolis force and the second order angular term R_{ce} describes the centrifugal force in the rotation with $\gamma \in \mathbb{R}$ the angular velocity/rotating speed. They are the two inertial forces giving by the change of frame to rotate with the background environment. The parameter c is the speed of light, $\hbar$ is the Planck constant, m denotes the mass and $\lambda \in \mathbb{R}$ denotes the strength of the self-interaction in the system. In the so-called nonrelativistic limit — i.e. $c \rightarrow \infty$, as shown mathematically in [25], the limit of RKG equation is consistent with the rotating nonlinear Schrödinger (RNLS) equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + \lambda|\psi|^2\psi + \gamma L_z\psi, \quad (1.2)$$

which is the key model for rotational Bose-Einstein condensates [1, 8]. The centrifugal term R_{ce} in the limit $c \rightarrow \infty$ is a higher order term [25] and this leaves only the contribution from the Coriolis term in the above limit equation (1.2). Thus, the RKG equation which is able to cover the entire velocity range, is considered as a more comprehensive model [25, 35] and can be used to describe rotational relativistic superfluid in general.

Numerically, the RNLS model (1.2) has been extensively studied and understood in the literature. For numerical investigations of its stationary states, we refer to [2–4, 8, 20]