East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2024-146.021224

Vol. **15**, No. 3, pp. 650-668
August 2025

Local Ultraconvergence of Quadratic Rectangular Element

Wenming He[1], Mingxiang Deng[2,*], Yongping Feng[2] and Xiaofei Guan[3]

[1]*Department of Statistics and Mathematics, Guangdong University of Finance & Economics, Guangzhou 510320, China.*
[2]*School of Mathematics and Information of Science, Guangzhou University, Guangzhou 510006, China.*
[3]*School of Mathematical Sciences, Tongji University, Shanghai 200092, China.*

Received 28 August 2024; Accepted (in revised version) 2 December 2024.

Abstract. A state of the art technology is employed to investigate the local ultraconvergence properties of a quadratic rectangular element for the Poisson equation. The proposed method combine advantages of a novel interpolation postprocessing operator $\overline{P}^{6}_{6h,m}R^{*}_{h}$, the Richardson extrapolation technique, and properties of a discrete Green's function. The local ultraconvergence of the post-processed gradient of the finite element solution is derived with the order $\mathscr{O}(h^6|\ln h|^2)$. A numerical example shows a good agreement with the theoretical findings.

AMS subject classifications: 65N30, 45N08

Key words: Ultraconvergence, quadratic rectangular element, integral identity, local symmetric, interpolation postprocessing.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ a bounded rectangular domain. In this work, we study the local ultraconvergence of the post-processed gradient of the quadratic rectangular finite element approximation for the following Poisson problem with the Dirichlet boundary condition:

$$\begin{aligned} &\mathscr{L}v(x) \equiv -\Delta v(x) = f(x) && \text{in } \Omega, \\ &v(x) = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where f is a sufficiently smooth function.

The natural superconvergence/ultraconvergence in the finite element method (FEM) is an important phenomenon. Since 1970s, the natural superconvergence/ultraconvergence

*Corresponding author. *Email addresses:* `he_wenming@aliyun.com` (W. He), `dmx@gzhu.edu.cn` (M. Deng), `fypmath@gzhu.edu.cn` (Y. Feng), `guanxf@tongji.edu.cn` (X. Guan)

in classical conforming finite element has been extensively studied in Refs. [1, 7, 9, 10, 12–18, 20, 21, 23]. Recently, the natural superconvergence/ultraconvergence of other finite element methods have been studied. In this article, we explore the natural superconvergence/ultraconvergence of a classic conforming finite element method. We believe that this research contributes significantly to the understanding of inherent superconvergence/ultraconvergence of other finite element methods such as C^1 conforming Petrov-Galerkin and discontinuous Galerkin methods.

We start with recalling superconvergence/ ultraconvergence results for the classic conforming quadratic rectangular finite element method. Let $\mathscr{T}_h$ be a uniform rectangular mesh over Ω and $\mathscr{T}_{h/2}$ be the mesh obtained from $\mathscr{T}_h$ by dividing it into four equal meshes. Assume that $R_h v(x)$ is the finite element approximation of $v(x)$ by bi-quadratic piecewise polynomials on $\mathscr{T}_h$. Zhu [23] derived the following superconvergence estimate:

$$h\|I_h v - R_h v\|_{W^{1,\infty}(\Omega)} + \|I_h v - R_h v\|_{L^\infty(\Omega)} \le ch^4|\ln h|\|v\|_{W^{4,\infty}(\Omega)}.$$

Besides, Q. Lin and J. Lin [12] applied the Richardson extrapolation technique to investigate the ultraconvergence of quadratic rectangular finite element method for the problem (1.1). In particular, they showed that

$$\|v - R_h^* v\|_{L^\infty(\Omega)} \le ch^6|\ln h|\|v\|_{W^{6,\infty}(\Omega)}, \tag{1.2}$$

where

$$R_h^* v(x) = \frac{16R_{h/2}v(x) - R_h v(x)}{15}. \tag{1.3}$$

The interpolation is another important post-processing technique to produce high accuracy gradient. It plays an important role in mechanical engineering [10, 20, 23]. Schatz and Wahlbin [14] developed a symmetry theory for finite element method. In particular, they observed that for meshes with a local symmetry, the corresponding finite element solution has natural superconvergence properties. Using piecewise polynomials of even degree $k \ge 2$, they showed that at local symmetric vertex x_0, the finite element solutions converge to the exact solution with the rate $\mathcal{O}(h^{k+2-\varepsilon})$, where $\varepsilon > 0$ is an arbitrary small positive number. If k is odd, the discrete gradient of the finite element solution converges to the gradient of the exact solution with the rate $\mathcal{O}(h^{k+1-\varepsilon})$. Note that a vertex x_0 is called local symmetric if there is an $r > 0$ such that in the neighborhood $B(x_0, r) = \{y : |x_0 - y| \le r\}$ of x_0, the underlying mesh is symmetric.

In this paper, we develop a new state of the art post-processing method, which preserves excellent properties of the Richardson extrapolation and interpolation and provides a higher-order gradient approximation. Furthermore, we approximate the derivative $\partial R_h^* v(x_0)/\partial x_m$ by the term $\overline{P}^6_{6h,m} R_h^* v(x_0)$, (3.3) below, constructed by using the Richardson extrapolation and interpolation post-processing technique. Note that $x_0 \in \mathscr{N}_h$ is an interior vertex so that $B(x_0, 2\kappa) \subset \tau \subset \Omega$, where κ is a positive constant independent of h and v, and $\mathscr{N}_h$ is a set of all vertices of $\mathscr{T}_h$. While proving the ultraconvergence property of $\overline{P}^6_{6h,m} R_h^* v(x_0)$, we also study the discrete Green's function. The translation invariance of