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Stochastic Symplectic Exponential Runge-Kutta Integrators for Semilinear SDEs and Applications to Stochastic Nonlinear Schrödinger Equation

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Abstract. Symplecticity is a significant property of stochastic Hamiltonian systems, and the symplectic methods are very attractive. Compared with the classical non-exponential Runge-Kutta methods, the exponential Runge-Kutta methods are more suitable for stiff problems. Therefore, the focus of this paper is on constructing stochastic symplectic exponential Runge-Kutta (SSERK) integrators for semilinear stochastic differential equations (SDEs) driven by multiplicative noise. The first is to establish the symplectic conditions of stochastic exponential Runge-Kutta (SERK) methods. It can be found that when the stiffness matrix is $\mathbf{0}$, these conditions will degenerate into the symplectic conditions of classical stochastic Runge-Kutta methods. Based on this idea, we construct a class of SSERK integrators with remarkable properties of structure-preservation. In addition, we verify the existence of the quadratic first integral of the stochastic Hamiltonian system and investigate the connection between preserving both the quadratic first integral and the symplectic structure by the SERK methods. Numerical experiments demonstrate a better structure-preserving ability and a higher accuracy of the SSERK integrators in solving the considered semilinear SDEs than the corresponding stochastic symplectic Runge-Kutta integrators. Excitingly, the SSERK integrators perform well when applied to the temporal discretization of stochastic nonlinear Schrödinger equation.

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Key words: Semilinear stochastic differential equations, stochastic symplectic exponential Runge-Kutta integrators, symplecticity, quadratic first integral, stochastic nonlinear Schrödinger equation.

1. Introduction

Stochastic differential equations enhance the fidelity of modeling real-world systems subject to inherent randomness and uncertainty. Their widespread applicability spans diverse fields such as physics, engineering, chemistry, finance, biology and complex networks

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under uncertain conditions. The inclusion of stochastic components in SDEs introduces computational and analytical challenges. Analytical solutions are often elusive owing to the complexity induced by randomness. The significance of studying the efficient numerical methods for solving SDEs is unequivocal.

In this article, we mainly explore structure-preserving algorithms for semilinear SDEs in the Stratonovich sense below

$$\mathrm{d}\boldsymbol{Z}(t)=\big(A\boldsymbol{Z}(t)+G_0(\boldsymbol{Z}(t))\big)\mathrm{d}t+\sum_{m=1}^{\mathscr{M}}G_m\big(\boldsymbol{Z}(t)\big)\circ\mathrm{d}W_m(t),\quad \boldsymbol{Z}(0)=\boldsymbol{Z}_0\in\mathbb{R}^{\bar{d}},\tag{1.1}$$

where the drift coefficients containing linear term $A\boldsymbol{Z}(t)$ and nonlinear term $G_0(\boldsymbol{Z}(t))$, the $G_m(\boldsymbol{Z}(t))$, $m=1,2,\ldots,\mathscr{M}$ in the diffusion term are nonlinear functions of $\boldsymbol{Z}(t)$. Here $t\in[0,T]$, A is a $\bar{d}\times\bar{d}$ constant matrix with eigenvalues of large modulus, and it can be expressed as $A=J^{-1}Q$, where

$$J=\begin{pmatrix}0 & -\mathrm{I}_d\\ \mathrm{I}_d & 0\end{pmatrix}$$

is a canonical symplectic matrix with $d\times d$ identity matrix I_d ($2d=\bar{d}$), and Q is a symmetric matrix. $W_m(t)$, $m=1,2,\ldots,\mathscr{M}$ are all standard Wiener processes, which are mutually independent and defined on a complete probability space $(\Omega,\mathscr{F},P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$.

The stiffness caused by matrix A will make it difficult to accurately simulate system (1.1) for a long time [18,37]. A wide variety of exponential methods have been proposed to replace traditional non-exponential methods to address the challenges posed by stiff problems. So far, much effort has been made in designing various exponential methods for solving ordinary differential equations (ODEs) [4,21,22,24,28]. With the development of numerical methods for SDEs, a series of stochastic exponential methods with different advantages have also been proposed [2,8,9,27,34,36]. Related research involves the construction, convergence, and stability of the methods. In addition, the damped Hamiltonian systems are a type of systems with an important physical background, and the exponential methods are proposed to preserve conformal symplectic structure [4,27,34,35], thereby achieving higher calculation accuracy. In order to specifically deal with second-order highly oscillatory problems, various extended Rung-Kutta-Nyström (ERKN) methods have been developed [11,20,31], which can also be classified as exponential methods. The system (1.1) will become a stochastic Hamiltonian system if we let $\boldsymbol{Z}(t)=(p(t),q(t))^{\mathrm{T}}$ and $G_m(\boldsymbol{Z}(t))=J^{-1}\nabla H_m(\boldsymbol{Z}(t))$, $m=0,1,\ldots,\mathscr{M}$. It can be written as

$$\mathrm{d}\boldsymbol{Z}(t)=\big(A\boldsymbol{Z}(t)+J^{-1}\nabla H_0(\boldsymbol{Z}(t))\big)\,\mathrm{d}t+\sum_{m=1}^{\mathscr{M}}J^{-1}\nabla H_m\big(\boldsymbol{Z}(t)\big)\circ\mathrm{d}W_m(t),\tag{1.2}$$

where H_m, $m=0,\ldots,\mathscr{M}$ are smooth Hamiltonian functions, and the initial value $\boldsymbol{Z}(0)=(p_0,q_0)^{\mathrm{T}}$. Here, $p(t)$ and $q(t)$ are column vectors with d-dimensional, p_0 and q_0 are the value of $p(t)$ and $q(t)$ at $t=0$, respectively. When applied to Hamiltonian systems, symplectic methods possess obvious advantages in structure-preserving and stability for a long time [10,13,25,32,38]. Taking into account the stiffness and symplecticity of the stochastic Hamiltonian system (1.2), we establish and analyze symplectic conditions of stochastic