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A Novel Nonconvex Rank Approximation with Application to the Matrix Completion

Jin-Liang Xiao, Ting-Zhu Huang*, Zhong-Cheng Wu and Liang-Jian Deng

School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, P.R. China.

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Abstract. The matrix rank approximation has shown high effectiveness in the matrix rank minimization (MRM) problem, which aims to recover the underlying low-rank structure from the observed matrix by imposing the rank constraint. The nuclear norm, serving as a convex surrogate of matrix rank, is employed in the MRM problem by shrinking singular values of the observed entry. However, this substitution treats each singular value equally, which is virtually ℓ_1-norm penalty of the singular value vector. Theoretically, the rank function of the matrix can be considered as ℓ_0-norm of its singular values. Consequently, minimizing the nuclear norm frequently results in biased solutions in various applications. In this article, we first propose a novel nonconvex rank approximation, named tight and flexible rank (TFR) approximation, to describe rank function effectively. Specifically, the TFR approximation can more tightly approach the rank function and exhibit greater flexibility in handling diverse singular values, as compared to existing nonconvex rank approximations. Furthermore, we apply TFR approximation to matrix completion and develop a solving algorithm with guaranteed convergence based on the framework of proximal alternating minimization. Extensive experiments reveal that the proposed matrix completion model with TFR approximation outperforms several existing state-of-the-art convex and nonconvex methods.

AMS subject classifications: 68U10, 65K10, 15A83

Key words: Rank minimization, low-rank matrix completion, rank approximation.

1. Introduction

Matrices are widely used in various fields, including computer vision and machine learning [10, 27, 43–45, 47, 60, 61], where some of their features, especially the low-rank property, can be utilized [36, 39, 56, 63]. It is worth noting that many related tasks — e.g. matrix completion [5], compressive sensing [11], and image denoising [13, 38, 70], can be

*Corresponding author. *Email addresses:* jinliang_xiao@163.com (J.-L. Xiao), tingzhuhuang@126.com (T.-Z. Huang), wuzhch97@163.com (Z.-C. Wu), liangjian.deng@uestc.edu.cn (L.-J. Deng)

described as the matrix rank minimization (MRM) problem, which consists in finding an approximate low-rank matrix from its degraded observation by the rank constraint [29]. Mathematically, this rank minimization problem can be formulated as follows:

$$\begin{aligned} &\min_{\mathbf{X}\in\mathbb{R}^{m\times n}} \ \operatorname{rank}(\mathbf{X}) \\ &\quad \text{s.t.} \ \ \mathscr{A}(\mathbf{X}) = \mathbf{B}, \end{aligned} \tag{1.1}$$

where $\mathbf{X} \in \mathbb{R}^{m\times n}$ is the underlying matrix, $\mathscr{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ a linear map, and $\mathbf{B} \in \mathbb{R}^{m\times n}$ the observed matrix. The choice of $\mathscr{A}$ depends on the specific application. Since the direct solution of the problem (1.1) is NP-hard [6], it usually depends on replacing the discontinuous rank function by an appropriate matrix rank approximation [20,21,62]. The later can be represented as

$$\begin{aligned} &\min_{\mathbf{X}\in\mathbb{R}^{m\times n}} \ \Psi(\mathbf{X}) \\ &\quad \text{s.t.} \ \ \mathscr{A}(\mathbf{X}) = \mathbf{B}, \end{aligned} \tag{1.2}$$

where $\Psi(\mathbf{X})$ is the rank approximation of matrix $\mathbf{X}$. Because the matrix rank is the number of non-zero singular values — i.e. ℓ_0-norm of the singular value vector, $\Psi(\mathbf{X})$ is usually defined as a function of singular values [24,71]. Thus, the MRM problem (1.1) is often approached by minimizing an appropriate rank approximation that penalizes singular values. This method is widely-used in various applications [40,48,65].

As the tightest convex approximation of the matrix rank, the nuclear norm is defined as the sum of the singular values. This transforms the minimization of the matrix rank into a constraint on singular values of the underlying matrix [15]. Candès and Recht [7] proved that low-rank structures can be extracted from the degraded matrix by minimizing the nuclear norm with a high probability. The application of the nuclear norm further demonstrates the effectiveness of the rank approximation defined by singular values [25]. Virtually, the singular value provides quantifiable information of the matrix. For example, larger singular values usually contain significant information about textures and edges [59]. However, the nuclear norm treats the singular values of the matrix equally. As a result, the nuclear norm shrinks the same value for each singular value. Numerous studies — e.g. [33,50,68], show that usually such a uniform shrinkage leads to a restricted performance.

Theoretically, the rank function of the matrix is ℓ_0-norm of its singular values, and the relationship between the nuclear norm and the rank of matrices can be seen as the relationship between ℓ_0-norm and ℓ_1-norm of singular value vectors [37], cf. Section 2 for more details. Clearly, there exists a distance between two norms for the constraint on singular values, which limits the performance of the nuclear norm. Note that there are many nonconvex matrix rank approximations aimed to better describe the ℓ_0-norm for singular values, [35,40]. In particular, Hu *et al.* [24] proposed the truncated nuclear norm, defined as the sum of the smaller singular values. Dong *et al.* [11] achieved promising results on compressive sensing by using the nonconvex logdet function as the surrogate of the rank function. Kang *et al.* [26] successfully applied the logdet function to recommender system via matrix completion. Nie *et al.* [34] utilized the Schatten p-norm for low-rank matrix restoration. Chen *et al.* [8] proposed the logarithmic norm to induce a sparsity-driven