

Convergence of a Discontinuous Galerkin Method on Bakhvalov-Type Meshes for Singularly Perturbed Volterra Integro-Differential Equations

Yige Liao and Xianbing Luo*

School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China.

Received 3 May 2024; Accepted (in revised version) 14 August 2024.

Abstract. A discontinuous Galerkin (DG) method on Bakhvalov-type (B -type) meshes for singularly perturbed Volterra integro-differential equations (SPVIDEs) is proposed. We derive abstract error bounds of the DG method for the SPVIDEs in the L^2 -norm. It is shown that the approximate solution generated by the DG method on B -type meshes has optimal convergence rate $k + 1$ in the L^2 -norm, when using the piecewise polynomial space of degree k . Numerical simulations demonstrate the validity of the theoretical results.

AMS subject classifications: 65L20, 65L50, 65L70

Key words: Singularly perturbed, Bakhvalov mesh, discontinuous Galerkin, parameter-uniform convergence.

1. Introduction

In this work, we study a DG method for the SPVIDEs

$$\begin{aligned} \varepsilon y'(x) + c(x)y(x) + \int_0^x L(x, t)y(t)dt &= g(x), \quad x \in \Omega := (0, 1], \\ y(0) &= y_0 \in R, \end{aligned} \tag{1.1}$$

where ε is a small perturbation parameter such that $0 < \varepsilon \ll 1$ and $c(x)$, $g(x)$, $L(x, t)$ are sufficiently smooth functions. Meanwhile, we assume that there are two positive constants $\beta, \bar{L}$ such that

$$c(x) \geq \beta > 0, \quad |L(x, t)| \leq \bar{L}, \quad x, t \in [0, 1].$$

Under these conditions, the problem (1.1) has a unique solution y , cf. [1]. Singularly

*Corresponding author. Email addresses: lyg199600@163.com (Y. Liao), xbluo1@gzu.edu.cn (X. Luo)

perturbed problem (SPP) often arises in such fields as physical and epidemic dynamics, and biological systems [6, 9, 14].

When ε tends to 0, the solution y of such equations has an exponential boundary layer of width $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$ at $x = 0$, which complicates the development of an effective numerical technique. Therefore, in recent years, special numerical methods have been developed to deal with SPVIDE [7, 8, 15, 19, 20, 26]. In particular, Yapman and Amiraliyev [26] designed a second-order fitted difference method on a piecewise-uniform Shishkin (puS-) mesh. Moreover, Long *et al.* [15] utilized the Richardson extrapolation technique to enhance the pointwise accuracy of the upwind difference scheme from the first-order to the second-order.

The aforementioned methods primarily focus on finite difference methods, which are usually low order convergent. Since DG methods have a great flexibility for implementation and high-order accuracy, they are widely applied to initial-value problems [5, 10] and Volterra integral and integro-differential equations [22, 23].

Lately, a host of researchers have successfully used the DG method to numerically solve SPPs — cf. [13, 21, 24, 25, 27] and references therein. For instance, Xie and Zhang [24, 25] employed a DG method to one-dimensional SPPs. They achieved the $\mathcal{O}((N^{-1} \ln N)^{k+1})$ convergence in the L^2 -norm on a puS-mesh, where N represent the number of mesh subintervals and k denote the degree of the approximate solution. For a fourth-order SPP, the error behaviour of a local DG (LDG) method discussed by Liu and Cheng [13]. They proved that the LDG scheme achieves optimal convergence in the energy norm on a puS-mesh. Especially, Tao and Xie [21] designed a DG scheme on a puS-mesh for the problem (1.1) and established $\mathcal{O}((N^{-1} \ln N)^{k+1})$ convergence in the L^2 -norm.

Up to now, the convergence has been mainly studied for DG methods on the puS-mesh because such a mesh greatly simplifies the corresponding analysis. However, their drawback is that they produce errors which contains $\ln N$ -factors. Noting that B -type meshes do not suffer from this, Zhang *et al.* [27] rigorously analysed the errors of local DG methods on B -type meshes for SPPs.

To the best of our knowledge, there are no works devoted to the DG methods for problem (1.1) on B -type meshes. Therefore, here we study the convergence of the DG method on B -type meshes in the L^2 -norm. Furthermore, we also analyse L^2 -error bound of the DG method on Shishkin-type (S -type) meshes that consist of the puS-mesh and its revised versions.

The outline of the paper is as follows. The construction and some corresponding properties of B -type meshes are presented Section 2. In Section 3, we consider a DG scheme with upwind flux and show that the numerical scheme has a unique solution. A complete ε -uniform convergence analysis is reported in Section 4. Theoretical findings are illustrated by the results of numerical experiments in Section 5. Some concluding remarks are given in Section 6.

Notation. Throughout this article, we use C to denote a generic positive constant that unaffected by ε and the discretization parameter N . For $D \subseteq \Omega$, we use the conventional symbols for Hilbert spaces $L^p(D)$. To simplify, we replace $\|\cdot\|_{L^2(D)}$ by $\|\cdot\|_D$ and omit the subscript D from the previously mentioned norms, when $D = \Omega$.