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The Optimal Production Transport: Model and Algorithm

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Abstract. We propose the optimal production transport model, which is an extension of the classical optimal transport model. We observe in economics, the production of the factories can always be adjusted within a certain range, while the classical optimal transport does not take this situation into account. Therefore, differing from the classical optimal transport, one of the marginals is allowed to vary within a certain range in our proposed model. To address this, we introduce a multiple relaxation optimal production transport model and propose the generalized alternating Sinkhorn algorithms, inspired by the Sinkhorn algorithm and the double regularization method. By incorporating multiple relaxation variables and multiple regularization terms, the inequality and capacity constraints in the optimal production transport model are naturally satisfied. Alternating iteration algorithms are derived based on the duality of the regularized model. We also provide a theoretical analysis to guarantee the convergence of our proposed algorithms. Numerical results indicate significant advantages in terms of accuracy and efficiency. Furthermore, we apply the optimal production transport model to the coal production and transport problem. Numerical simulation demonstrates that our proposed model can save the production and transport cost by 13.17%.

AMS subject classifications: 49M25, 65K10

Key words: Optimal production and transport, multiple relaxation, generalized alternating Sinkhorn algorithm, supply chain.

1. Introduction

The widely studied optimal transport (OT) theory can be traced back to 1781 in France, where Monge first formalized it within the field of civil engineering [27]. Major advances in the OT theory were made by Soviet mathematician and economist Kantorovich [17,38]. He introduced a relaxation technique to transform the OT problem into a linear programming problem and further provided an economic interpretation [8], that is the optimal

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allocation and utilization of resources in the whole society. In 1975, the Nobel Memorial Prize in Economic Sciences, which he shared with Koopmans, was given “for their contributions to the theory of optimum allocation of resources”. Subsequently, OT has been widely generalized and successfully applied in economics, including the classical discrete choice model [6, 14], the partial identification with random sets [12, 13], the hedonic equilibrium problem [10, 26], and the price discrimination and implementability [23], among others.

In the classical OT problem, researchers consider how to transport goods between fixed production and consumption, in other words, to obtain the optimal transport plan. However, in economics, the production between different factories always can be adjusted within a certain range, and various factories may have different costs to produce one unit of goods [19, 24, 34]. This situation requires us to find the optimal production transport plan simultaneously with the minimum total cost, i.e., the summation of transport cost and production cost. To the best of our knowledge, this type of extension has not been considered in existing OT models. Therefore, it is essential to explore the formulation of this variation.

Remark 1.1. There are extensions of the classical OT [33] assuming that the marginal distribution is restricted within two boundaries instead of a fixed one. Although such extensions give more freedom for the transport to find solutions, they do not impose restrictions on the transport plan, which is inapplicable to real-world cases involving capacity constraints. It thus does not address the focused issue in this paper.

In this paper, we generalize the classical OT model to an optimal production transport (OPT) model. This allows the marginal to vary within a certain range. In particular, the OPT model can be expressed as

$$\begin{aligned} \min_{\gamma,u} \quad & \int_{\Omega_1\times\Omega_2} C(x,y)\gamma(x,y)dxdy + \int_{\Omega_1} p(x)u(x)dx, \\ \text{s.t.} \quad & \int_{\Omega_2} \gamma(x,y)dy = u(x), \quad \int_{\Omega_1} \gamma(x,y)dx = v(y), \\ & \theta(x,y) \le \gamma(x,y) \le \eta(x,y), \quad \hat{u}(x) \le u(x) \le \overline{u}(x), \end{aligned} \tag{1.1}$$

where C and p respectively denote the transport production costs, γ is the transport plan with the lower bound θ and the upper bound η, and u represents the production plan with the lower bound $\hat{u}$ and the upper bound $\overline{u}$. Note that when $\theta = 0$, $\eta = +\infty$, and $\hat{u} = \overline{u}$, this model degenerates into the classic OT model.

Although there exists a large number of algorithms designed for the classical OT model [3, 31], the OPT model brings out new difficulties, which hinder applying these algorithms to the OPT model directly. More specifically, the primary difficulties arise in two aspects — viz.

(1) The objective function contains an additional term of a variable marginal compared with the classical OT, and the duality is in a max-min form when directly applying alternating algorithms for the classical OT.