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A Generalized Quasi-Boundary Value Method for an Inverse Source Problem in a Distributed Order Time-Fractional Diffusion Equation

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Abstract. Subdiffusion equations with distributed-order fractional derivatives describe important physical phenomena. In this paper, we consider an inverse space-dependent source term problem governed by a distributed order time-fractional diffusion equation using final time data. Based on the series expression of the solution, the inverse source problem can be transformed into a first kind of Fredholm integral equation. The existence, uniqueness and a conditional stability of the considered inverse problem are established. Building upon this foundation, a generalized quasi-boundary value regularization method is proposed to solve the inverse source problem, and then we prove the well-posedness of the regularized problem. Further, we provide the convergence rates of the regularized solution by employing an a priori and an a posteriori regularization parameter choice rule. Numerical examples in one-dimensional and two-dimensional cases are provided to validate the effectiveness of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Inverse source problem, distributed order time-fractional diffusion equation, generalized quasi-boundary value method, convergence rate, numerical experiment.

1. Introduction

In recent years, the distributed order fractional derivative has gained wide attention across various fields, such as dynamic transport [4, 6, 29, 35], option pricing [17, 31], viscoelastic materials [18], and control theory [45], due to its effectiveness in modeling complex multiscale phenomena. While single- and multi-term fractional partial differential equations [1,22] might inadequately capture certain intricate physical processes such as diffusion in multifractal media, the distributed derivative provides a remedy. The distributed derivative is derived by modifying the weighted integral of fractional order derivative over a fixed range, and subsequently applied to distributed order partial differential equations. It is worth noting that the distributed order time-fractional diffusion equation accurately

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characterizes ultra-low anomalous diffusion with logarithmic growth in mean square displacement [4,6,12,34,35].

In this paper, we study an inverse source problem for a distributed order diffusion equation. Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we consider the following distributed order time-fractional diffusion equation:

$$\begin{aligned} &\mathbb{D}_t^{(\omega)}u(x,t)+Lu(x,t)=f(x)q(t), && (x,t)\in\Omega\times(0,T],\\ &u(x,t)=0, && (x,t)\in\partial\Omega\times(0,T],\\ &u(x,0)=\phi(x), && x\in\Omega, \end{aligned} \tag{1.1}$$

where a distributed order time-fractional derivative $\mathbb{D}_t^{(\omega)}u(x,t)$ is defined as

$$\mathbb{D}_t^{(\omega)}u(x,t)=\int_0^1 \partial_t^{\alpha}u(x,t)\omega(\alpha)d\alpha, \tag{1.2}$$

where ∂_t^{α} is the Caputo fractional derivative defined by

$$\partial_t^{\alpha}u(x,t)=\begin{cases} u(x,t), & \alpha=0,\\ \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-\tau)^{-\alpha}\frac{\partial u}{\partial\tau}(x,\tau)d\tau, & 0<\alpha<1,\\ \dfrac{\partial u}{\partial t}(x,t), & \alpha=1, \end{cases}$$

and $\omega:[0,1]\to\mathbb{R}$ is a weight function. In this paper, we assume that ω satisfies

$$\omega\in C[0,1],\quad \omega(\alpha)\geq 0,\quad \omega\not\equiv 0,\quad \int_0^1\omega(\alpha)d\alpha=c_0>0, \tag{1.3}$$

and L is a symmetric uniformly elliptic operator considered on $D(L)=H^2(\Omega)\cap H_0^1(\Omega)$ and defined by

$$Lu(x,t)=-\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d}a_{ij}(x)\frac{\partial}{\partial x_j}u(x,t)\right)+c(x)u(x,t),\quad x\in\Omega$$

with the coefficients satisfying

$$\begin{aligned} &a_{ij}=a_{ji},\quad a_{ij}\in C^{\infty}(\bar{\Omega}),\qquad 1\leq i,j\leq d,\\ &c(x)\geq 0,\quad c(x)\in C^{\infty}(\bar{\Omega}),\quad x\in\bar{\Omega},\\ &\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\xi_j\geq \nu\sum_{i=1}^{d}\xi_i^2,\quad x\in\bar{\Omega},\quad \xi\in\mathbb{R}^d,\ \text{for a constant } \nu>0. \end{aligned}$$

The direct problem for distributed order fractional diffusion equations has been investigated by various researchers. Some results on the weak solution expression are provided