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Crank-Nicolson ADI Finite Difference Method for Three-Dimensional Nonlinear Partial Integro-Differential Equations with Weak Singular Kernels

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Abstract. The main objective of this study is to present a fast and efficient numerical scheme for solving nonlinear integral differential equations with weak singular kernels in three-dimensional domain. First, the temporal derivative and integral term are approximated by the Crank-Nicolson (CN) method and the second-order fractional quadrature rule. After that the spatial discretization is carried out by combining the finite difference method and the alternating direction implicit (ADI) method, and the nonlinear term are approximated using the Taylor expansion. The stability and convergence of the proposed scheme are analyzed, followed by the verification of the theoretical results through numerical experiments.

AMS subject classifications: 35R09, 35R11, 65M06, 65M12
Key words: Three-dimensional nonlinear integral differential equation, alternating direction implicit method, finite difference method, stability, convergence.

1. Introduction

In this paper, we present the CN ADI finite difference method for solving the following three-dimensional nonlinear integral-differential equations with a weakly singular kernel:

$$\frac{\partial u}{\partial t} - \kappa\Delta u - I^{(\alpha)}\Delta u = f(x,y,z,t) + \tilde{r}(u), \quad (x,y,z)\in\Omega, \quad t\in(0,T] \tag{1.1}$$

with the boundary and initial conditions

$$u(x,y,z,t) = 0, \quad (x,y,z)\in\partial\Omega, \quad t\in(0,T], \tag{1.2}$$

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$$u(x,y,z,0)=\varphi(x,y,z), \quad (x,y,z)\in\bar{\Omega}, \tag{1.3}$$

where $\alpha\in(0,1)$, $\kappa>0$, $\partial\Omega$ is the boundary of the domain $\Omega=(0,H_x)\times(0,H_y)\times(0,H_z)$, $\overline{\Omega}=\Omega\cup\partial\Omega$, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ is the three-dimensional Laplace operator, and $f(x,y,z,t)$, $\varphi(x,y,z)$ are given functions. We also assume that term $\tilde{r}(u)$ in (1.1) is a continuous function satisfying the Lipschitz condition

$$|\tilde{r}(u_1)-\tilde{r}(u_2)|\le L|u_1-u_2|, \tag{1.4}$$

where $\tilde{r}(0)=0$ and L is the Lipschitz constant [1]. Besides, $I^{(\alpha)}$, $0<\alpha<1$ is the Riemann-Liouville (R-L) integral defined by

$$I^{(\alpha)}v(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds, \quad t>0. \tag{1.5}$$

It is well known that many phenomena in engineering, physics, and chemistry are modeled by using the fractional calculus — i.e. the derivatives and integrals of arbitrary order [7, 8, 15, 23], including fractional integro-differential equations with weakly singular kernels. In particular, the Eq. (1.1) arises in elasticity and fracture mechanics [29], radiative equilibrium [9], and heat conduction problems [22]. Analytical solutions to such problems are usually available neither easily nor conveniently, so it is important to obtain their approximate solutions. This motivates us to develop numerical methods that are more efficient and accurate.

In recent years, various numerical methods for partial integro-differential equations with weakly singular kernels have been studied. Let us note the finite element method [10, 14], finite difference method [2, 3, 16, 26], orthogonal spline collocation method [17], and spectral method [27] used to solve one- and two-dimensional problems. On the other hand, three-dimensional nonlinear integral differential equations with a weakly singular kernel and their numerical solutions received less attention. This is also a motivation behind the present work. Moreover, ADI algorithms have been rarely applied nonlinear integral differential equations with a weak singular kernel. Therefore, here we aim to solve the three-dimensional nonlinear integral differential equations with a weakly singular kernel by a finite difference method. To achieve this, we design relevant ADI algorithms. The main advantage of such algorithms is the dividing the solution of a multidimensional problem into independent one-dimensional problems enhancing the computational efficiency compared to stable implicit schemes.

The main contributions of this paper can be summarized as follows:

(i) We propose an ADI finite difference method for numerically solving three-dimensional nonlinear integral-differential equations with a weakly singular kernel. Meanwhile, we provide three-dimensional numerical examples to demonstrate the effectiveness of the proposed scheme.

(ii) We expand the linear three-dimensional case from [18] to the three-dimensional nonlinear case, which enhances the complexity of both numerical construction and theoretical analysis.