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ON THE CONVERGENCE OF QUASI-CONFORMING ELEMENTS FOR LINEAR ELASTICITY PROBLEMS*

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Abstract

Continuing the work in [1, 2], we discuss the convergence conditions and the error estimates of quasi-conforming elements for linear elasticity problems. Some results about curved elements for second-order boundary value problems are also given.

§1. Introduction

The quasi-conforming element method is very efficient and successful for elliptic problems (see [1—4]). The mathematical foundation of this method has been established in [1, 2], and several plate bending elements have been shown to be convergent. In the present paper, we mainly discuss the convergence conditions of quasi-conforming elements for linear elasticity problems and their error estimates. In addition we give some results about curved elements for second-order boundary value problems.

Let Ω be a bounded connected domain in R^n with Lipschitz-continuous boundary $\partial\Omega$. For each $v=(v_1, v_2, \cdots, v_n)$ in $(H^1(\Omega))^n$, we set

$$\begin{cases} \varepsilon_{ij}(v)=\varepsilon_{ji}(v)=(\partial_i v_j+\partial_j v_i)/2, & 1\leqslant i,\ j\leqslant n, \\ \sigma_{ij}(v)=\sigma_{ji}(v)=\lambda\left(\sum\limits_{l=1}^{n} \varepsilon_{ll}(v)\right)\delta_{ij}+2\mu\varepsilon_{ij}(v), & 1\leqslant i,\ j\leqslant n, \end{cases} \tag{1.1}$$

where δ_{ij} is Kronecker's symbol, and λ and μ are two positive constants.

Consider the boundary value problem of linear elasticity:

$$\begin{cases} -\sum\limits_{j=1}^{n} \partial_j\sigma_{ij}(u)=f_i, \quad i=1, \cdots, n, \text{ in } \Omega, \\ u|_{\partial\Omega}=0. \end{cases} \tag{1.2}$$

In order to discuss this problem, we need some notations. Let $H=\{\Lambda=(V_i, E_{ij})\,|\,V_i, E_{ij}\in L^2(\Omega), 1\leqslant i, j\leqslant n \text{ with } E_{ij}=E_{ji}\}$. For $v=(v_1, \cdots, v_n)$ in $(H^1(\Omega))^n$, define $Tv=(v_i, \varepsilon_{ij}(v))$. Then $T(H^1(\Omega))^n$ is a subspace of H. And for each Λ in H, define

$$\sigma_{ij}(\Lambda)=\sigma_{ji}(\Lambda)=\lambda\left(\sum_{l=1}^{n} E_{ll}\right)\delta_{ij}+2\mu E_{ij}, \quad 1\leqslant i,\ j\leqslant n. \tag{1.3}$$

We define the bilinear form $a(\cdot, \cdot)$ and the linear form $f(\cdot)$ on space H as

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follows:

$$a(\Lambda, \overline{\Lambda}) = \sum_{i,j=1}^{n} \int_{\Omega} \sigma_{ij}(\Lambda) \overline{E}_{ij} dx, \quad \forall \Lambda, \overline{\Lambda} \in H, \tag{1.4}$$

$$f(\Lambda) = \int_{\Omega} \sum_{i=1}^{n} f_i V_i dx, \quad \forall \Lambda \in H. \tag{1.5}$$

Then problem (1.2) is equivalent to the following variational problem:

$$u \in (H_0^1(\Omega))^n, \ a(Tu, Tv) = f(Tv), \quad \forall v \in (H_0^1(\Omega))^n. \tag{1.6}$$

Let $\{U_h\}$ be a family of finite dimensional subspaces in H for parameter h with $h>0$ and $h \to 0$. According to [1, 2], the finite element approximation with multiple sets of functions to problem (1.6) is the following problem

$$\Lambda_h^* \in U_h, \ a(\Lambda_h^*, \Lambda_h) = f(\Lambda_h), \quad \Lambda_h \in U_h. \tag{1.7}$$

The work in this paper is mainly devoted to the following subjects: i) the method for constructing spaces U_h using quasi-conforming elements, ii) the existence, uniqueness and convergence of the solution of problem (1.7).

§ 2. Quasi-Conforming Element Method

In this section we discuss how to construct spaces U_h by the quasi-conforming element method. From now on, we assume that Ω is a polyhedroid domain in R^n, and let K_h be a finite subdivision of Ω for each h with the properties K1 and K2:

K1. For every element K in K_h, K is an n-simplex (or n-parallelotope) and

$$\bigcup_{K \in K_h} = \overline{\Omega}.$$

K2. For every two different elements K and K' in K_h, $K \cap K'$ is an empty set or a common face of K and K'[1].

Let t be a positive number. For any n-simplex (or n-parallelotope) K, we give two linear interpolation operators Π_K: $H^t(K) \to L^2(K)$ and $\Pi_{\partial K}$: $H^t(K) \to L^2(\partial K)$ and some finite dimensional spaces consisting of polynomials, say N_K^{ij}, $1 \leqslant i, j \leqslant n$, with $N_K^{ij} = N_K^{ji}$. For each $v = (v_1, \cdots, v_n)$ in $(H^t(K))^n$, we define $E_K^{ij}(v)$ in N_K^{ij}, $1 \leqslant i, j \leqslant n$, by the following equations:

$$2\int_K pE_K^{ij}(v)\, dx = \int_{\partial K} p(\Pi_{\partial K} v_i N_j + \Pi_{\partial K} v_j N_i)\, ds$$
$$-\int_K (\partial_j p \Pi_K v_i + \partial_i p \Pi_K v_j)\, dx, \quad 1 \leqslant i, j \leqslant n, \ \forall p \in N_K^{ij}, \tag{2.1}$$

where $N = (N_1, \cdots, N_n)^T$ is the unit outward normal of ∂K. Then we use $\Pi_K v_j$ and $E_K^{ij}(v)$ to approximate v_j and $\varepsilon_{ij}(v)$ respectively.

Now we are in a position to construct spaces U_h. Define an operator Π_h: $(H^t(\Omega))^n \to H$ such that, for each v in $(H^t(\Omega))^n$, $\Pi_h v = (\Pi_h^j v, E_h^{ij} v)$ satisfies

$$\begin{cases} \Pi_h^j v|_K = \Pi_K(v_j|_K), & 1 \leqslant j \leqslant n, \ K \in K_h, \\ E_h^{ij} v|_K = E_K^{ij}(v|_K), & 1 \leqslant i, j \leqslant n, \ K \in K_h. \end{cases} \tag{2.2}$$

Then the spaces U_h are obtained by setting $U_h = \Pi_h(H^t(\Omega) \cap H_0^1(\Omega))^n$; this is the

1) A subset F is said to be a face of K if there exists a supporting hyper-plane $\mathscr{P}$ of K such that $F = K \cap \mathscr{P}$.

quasi-conforming element method.

We want to say some more about equations (2.1). Let $\phi_1, \cdots, \phi_M$ be a group of linearly independent functionals on $H^t(K)$, $G_1, \cdots, G_M$ a group of polynomials defined on K and $g_1, \cdots, g_M$ a group of functions in $L^2(\partial K)$. Assume that for each w in $H^t(K)$,

$$\Pi_K w=\sum_{j=1}^{M}\phi_j(w)G_j, \quad \Pi_{\partial K}w=\sum_{j=1}^{M}\phi_j(w)g_j. \tag{2.3}$$

Then for each v in $(H^t(K))^n$, denote

$$\phi(v)=(\phi_1(v_1), \cdots, \phi_M(v_1), \cdots, \phi_1(v_n), \cdots, \phi_M(v_n))^T.$$

Let L_{ij} be the dimension of N_K^{ij}, and choose a basis of N_K^{ij}. Denote the coordinate vector of $E_K^{ij}(v)$ with respect to this basis by $\beta_{ij}(v)$, and define

$$\beta(v)=(\beta_{11}^T(v), \cdots, \beta_{1n}^T(v), \beta_{22}^T(v), \cdots, \beta_{2n}^T(v), \cdots, \beta_{nn}^T(v))^T,$$

$$L=\sum_{1\le i\le j\le n}L_{ij}.$$

Then equations (2.1) become

$$A_K\beta(v)=Q_K\phi(v), \tag{2.4}$$

where A_K is a nonsingular $L\times L$ matrix and Q_K an $L\times nM$ matrix.

§ 3. Main Results

In order to discuss the convergence of quasi-conforming element method for equation (1.1), we need some more assumptions about the subdivision family $\{K_h\}$. Suppose that $\{K_h\}$ satisfies K1, K2 and the following condition K3:

K3. Let ρ_K be the greatest of diameters of all balls contained in K and h_K the greatest diameter of K; then there exist two positive constants η_1, η_2 independent of K and h, such that $h_K\leqslant\eta_1 h$ and $\eta_2 h\leqslant\rho_K$ are true for each K in K_h and each h.

On the space H, we define a semi-norm $|\cdot|_B$ as follows:

$$|A|_B^2=\int_\Omega\sum_{i,j=1}^{n}|E_{ij}|^2dx, \quad \forall A\in H. \tag{3.1}$$

We call $\{\Pi_K, \Pi_{\partial K}\}$ an affine family if there exists a fixed n-simplex (or n-parallelotope) $\hat{K}$ such that for each n-simplex K (or n-parallelotope), there exists an affine transformation $F_K\hat{x}=B_K\hat{x}+b_K$ satisfying $K=F_K\hat{K}$ and $\widehat{\Pi_K w}=\Pi_{\hat{K}}\hat{w}$ and $\widehat{\Pi_{\partial K}w}=\Pi_{\partial\hat{K}}\hat{w}$ for every w in $H^t(K)$, where B_K is a non-singular $n\times n$ matrix and b_K is a vector in R^n and if w is a function on K (or ∂K) then $\hat{w}(\hat{x})=w(F_K\hat{x})$ when $\hat{x}\in\hat{K}$ (or $\partial\hat{K}$) (see [5]).

Now we can give our main results.

Theorem. *Let conditions* K1—K3 *hold and assume that the following four statements are true:*

1) $\{\Pi_K, \Pi_{\partial K}\}$ is an affine family and there exists an integer $r_1\geqslant1$, such that $\Pi_{\hat{K}}$: $H^{r_1+1}(\hat{K})\to L^2(\hat{K})$ and $\Pi_{\partial\hat{K}}$: $H^{r_1+1}(\hat{K})\to L^2(\partial\hat{K})$ are continuous and for $\hat{p}$ in $P_{r_1}(\hat{K})$, $\Pi_{\hat{K}}\hat{p}=\hat{p}$, $\Pi_{\partial\hat{K}}\hat{p}=\hat{p}|_{\partial\hat{K}}$;

2) there exists a nonnegative integer r_2 and a space $N_{\hat{K}}$, such that, $N_{\hat{K}}\supset P_{r_2}(\hat{K})$ and $N_{\hat{K}}=\{\hat{p}|\hat{p}(\hat{x})=p(F_K\hat{x}), \forall p\in N_K^{ij}\}$ for $i, j=1, \cdots, n$;

3) *for each n-simplex (or n-parallelotope) K, the rank of the matrix Q_K is $nM-n(n+1)/2$;*

4) *there exists an integer $r_3 \geq 0$, such that for each w in $H^t(\Omega) \cap H_0^1(\Omega)$,*

$$p \in P_{r_3}(F), \quad \int_F p\Pi_{\partial K} w\, ds = \int_F p\Pi_{\partial K'} w\, ds$$

hold if F is an $(n-1)$-dimensional common face of both K and K' in K_h, and

$$p \in P_{r_3}(F), \quad \int_F p\Pi_{\partial K} w\, ds = 0$$

hold if F is an $(n-1)$-dimensional face of K with F on $\partial\Omega$, and $\Pi_{\partial K} w$ has the same values as w does at the vertices of K.

Then problem (1.7) has unique solutions $\Lambda_h^* = ((u_h^*)_i, (E_h^*)_{ij})$ for the spaces U_h, obtained by the quasi-conforming element method, and when u is in $(H^r(\Omega) \cap H_0^1(\Omega))^n$,

$$|Tu - \Lambda_h^*|_B \leq C\{h^{r_1}|u|_{r_1+1,\Omega} + h^{r_2+1}|u|_{r_2+2,\Omega} + h^{r_3+1}|u|_{r_3+2,\Omega}\} \tag{3.2}$$

holds for all h, where C is a constant independent of u and h and $r = \max\{r_1+1, r_2+2, r_3+2\}$, $|u|_{l,\Omega} = \left(\sum_{j=1}^{n} |u_j|_{l,\Omega}^2\right)^{1/2}$ for any integer l.

Using the Aubin–Nitsche technique, we can get the following conclusion (see [5, 6]).

Corollary. Let the conditions of the theorem hold. If Ω is convex and $\Pi_{\hat{K}}$: $H^2(\hat{K}) \to L^2(\hat{K})$ and $\Pi_{\partial\hat{K}}$: $H^2(\hat{K}) \to L^2(\partial\hat{K})$ are continuous, then there exists a constant C independent of h and u such that

$$\left(\sum_{j=1}^{n} \|u_j - (u_h^*)_j\|_{0,\Omega}^2\right)^{1/2} \leq C\{h^{r_1+1}|u|_{r_1+1,\Omega} + h^{r_2+2}|u|_{r_2+2,\Omega} + h^{r_3+2}|u|_{r_3+2,\Omega}\} \tag{3.3}$$

are uniformly true for all h.

The conditions of the theorem are not difficult to check and can be used as criteria for proving convergence and constructing elements. Mechanically, condition 3) implies that the states of the zero energy can only be the states of the rigid displacements, and condition 4) eliminates the rigid displacements of whole body Ω.

§ 4. The Proof of the Main Results

First, we prove some lemmas.

Lemma 1. *For each n-simplex (or n-parallelotope) K, there exists a constant $C(K)$ such that, for each v in $(H^t(K))^n$, the following inequality holds*

$$\sum_{i,j=1}^{n} |\partial_i \Pi_K v_j + \partial_j \Pi_K v_i|_{0,K}^2 + \sum_{j=1}^{n} |\Pi_K v_j - \Pi_{\partial K} v_j|_{0,\partial K}^2 \leq C(K) \sum_{i,j=1}^{n} |E_K^{ij}(v)|_{0,K}^2. \tag{4.1}$$

Proof. First, we show that if $E_K^{ij}(v) = 0$, $1 \leq i, j \leq n$, then $\partial_i \Pi_K v_j + \partial_j \Pi_K v_i = 0$ and $\Pi_{\partial K} v_j - \Pi_K v_j|_{\partial K} = 0$, $1 \leq i, j \leq n$. Because the rank of the matrix Q_K is $nM - n(n+1)/2$, the dimension of the solution space of the following equation is $n(n+1)/2$:

$$Q_K \phi = 0. \tag{4.2}$$

There exist $n(n+1)/2$ linearly independent vector functions v^l in $(P_1(K))^n$, by the theory of linear elasticity, such that

$$(\partial_j v_i^l + \partial_i v_j^l)/2 = 0, \quad 1 \leq i, j \leq n, \ l = 1, 2, \cdots, n(n+1)/2.$$

By condition 1) in the theorem, $\Pi_K v_j^l = v_j^l$ and $\Pi_{\partial K} v_j^l = v_j^l|_{\partial K}$; so $\phi(v^l)$ are the solutions of equation (4.2). What is more, $\phi(v^l)$, $l = 1, \cdots, n(n+1)/2$, are linearly independent; otherwise we can conclude that v^l are linearly dependent. Thus the solutions of equation (4.2) have the form

$$\phi = \sum_{l=1}^{n(n+1)/2} c_l \phi(v^l).$$

If $E_K^{ij}(v) = 0$ with $1 \leq i, j \leq n$, then $\phi(v)$ is a solution of equation (4.2), and for some numbers c_l,

$$\Pi_K v_j = \sum_{l=1}^{n(n+1)/2} c_l v_j^l, \quad \Pi_{\partial K} v_j = \sum_{l=1}^{n(n+1)/2} c_l v_j^l|_{\partial K}, \quad j = 1, \cdots, n.$$

Hence $\partial_i \Pi_K v_j + \partial_j \Pi_K v_i = 0$, $\Pi_{\partial K} v_j - \Pi_K v_j|_{\partial K} = 0$, $1 \leq i, j \leq n$. Thus $\left(\sum_{i,j=1}^{n} |E_K^{ij}(v)|_{0,K}^2\right)^{1/2}$ is a norm on space $\{(\Pi_K v_i - \Pi_{\partial K} v_i, \ \partial_i \Pi_K v_j + \partial_j \Pi_K v_i) \mid \forall v \in (H^t(K))^n\}$. It follows that inequality (4.1) is true.

Lemma 2. *There exists a constant $C > 0$ independent of K and h such that*

$$\sum_{i,j=1}^{n} |\partial_i \Pi_K v_j + \partial_j \Pi_K v_i|_{0,K} \leq C \sum_{i,j=1}^{n} |E_K^{ij}(v)|_{0,K}, \tag{4.3}$$

$$\sum_{j=1}^{n} |\Pi_K v_j - \Pi_{\partial K} v_j|_{0,\partial K} \leq C h^{1/2} \sum_{i,j=1}^{n} |E_K^{ij}(v)|_{0,K} \tag{4.4}$$

hold for all v in $(H^t(K))^n$, all K in K_h and all h.

Proof. Denote $J_\xi = \{K \mid$ the volume of K is equal to that of $\hat{K}$ and $h_K \leq \xi\}$ for any $\xi > 0$. We show that if J_ξ is not empty then there exists a constant $C(\xi)$ independent of K, such that

$$\sum_{j=1}^{n} |\Pi_K v_j - \Pi_{\partial K} v_j|_{0,\partial K}^2 + \sum_{i,j=1}^{n} |\partial_i \Pi_K v_j + \partial_j \Pi_K v_i|_{0,K}^2 \leq C(\xi) \sum_{i,j=1}^{n} |E_K^{ij}(v)|_{0,K}^2 \tag{4.5}$$

holds for v in $(H^t(K))^n$ and K in J_ξ. And this is equivalent to showing that

$$\sum_{j=1}^{n} |\Pi_{\hat{K}} \hat{v}_j - \Pi_{\partial \hat{K}} \hat{v}_j|_{0,\partial \hat{K}}^2 + \sum_{i,j=1}^{n} |\partial_i \widehat{\Pi_K v_j} + \partial_j \widehat{\Pi_K v_i}|_{0,\hat{K}}^2 \leq C(\xi) \sum_{i,j=1}^{n} |\widehat{E_K^{ij}(v)}|_{0,\hat{K}}^2 \tag{4.6}$$

holds for $\hat{v}$ in $(H^t(\hat{K}))^n$, where $v(x) = \hat{v}(F_K^{-1} x)$, $x \in K$.

Since $\{\Pi_K, \Pi_{\partial K}\}$ is an affine family, we can choose $\hat{\phi}_j(\hat{w})$ as the linear functionals $\phi_j(w)$ in (2.3) for every w in $H^t(K)$, i.e., set $G_j(x) = \hat{G}_j(F_K^{-1} x)$ and $g_j(x) = \hat{g}_j(F_K^{-1} x)$; then for every w in $H^t(K)$,

$$\Pi_K w = \sum_{j=1}^{M} \hat{\phi}_j(\hat{w}) G_j, \quad \Pi_{\partial K} w = \sum_{j=1}^{M} \hat{\phi}_j(\hat{w}) g_j.$$

Here for $\hat{w}$ in $H^t(\hat{K})$,

$$\Pi_{\hat{K}} \hat{w} = \sum_{j=1}^{M} \hat{\phi}_j(\hat{w}) \hat{G}_j, \quad \Pi_{\partial \hat{K}} \hat{w} = \sum_{j=1}^{M} \hat{\phi}_j(\hat{w}) \hat{g}_j.$$

In addition, we choose a basis of space $N_{\hat{K}}$ and let it be the basis for N_K^{ij}. For each v in $(H^t(K))^n$, denote

$$\hat{\phi}(\hat{v}) = (\hat{\phi}_1(\hat{v}_1), \cdots, \hat{\phi}_M(\hat{v}_1), \cdots, \hat{\phi}_1(\hat{v}_n), \cdots, \hat{\phi}_M(\hat{v}_n))^T.$$

Then equation (2.4) becomes

$$A_K \beta(v) = Q_K \hat{\phi}(\hat{v}),$$

where A_K and Q_K are continuous with respect to the matrix B_K of the affine transformation F_K. Let Ker Q_K be the solution space of the equation

$$Q_K\hat{\phi}=0.$$

Now we prove inequality (4.6). For this purpose, we assume that it does not hold. Then we can choose $\hat{v}^\nu$ in $(H^1(\hat{K}))^n$ and K_ν in J_t, for $\nu=1, 2, \cdots$, such that inequalities

$$\sum_{j=1}^{n}|\Pi_{\hat{K}}\hat{v}_j^\nu-\Pi_{\partial\hat{K}}\hat{v}_j^\nu|_{0,\partial\hat{K}}^2+\sum_{i,j=1}^{n}|\partial_i\widehat{\Pi_{K_\nu}v_j^\nu}+\partial_j\widehat{\Pi_{K_\nu}v_i^\nu}|_{0,\hat{K}}^2>\nu\sum_{i,j=1}^{n}|\widehat{E_{K_\nu}^{ij}(v^\nu)}|_{0,\hat{K}}^2 \quad (4.7)$$

hold. Because $\Pi_{\hat{K}}$, $\Pi_{\partial\hat{K}}$, $E_{K_\nu}^{ij}$, etc. are linear operators and

$$\Pi_{\partial\hat{K}}\hat{v}_j-\Pi_{\hat{K}}\hat{v}_j|_{\partial\hat{K}}=0,\ \partial_i\widehat{\Pi_{K_\nu}v_j}+\partial_j\widehat{\Pi_{K_\nu}v_i}=0,\ \widehat{E_{K_\nu}^{ij}(v)}=0,\quad 1\le i,\ j\le n,$$

when $\hat{\phi}(\hat{v})$ is in Ker Q_{K_ν}, we can choose $\hat{v}^\nu$ in (4.7) with the following properties:

$$\hat{\phi}(\hat{v}^\nu)\in(\mathrm{Ker}\ Q_{K_\nu})^\perp,\quad \|\hat{\phi}(\hat{v}^\nu)\|=1,\quad \nu=1, 2, \cdots, \quad (4.8)$$

where $\|\cdot\|$ is the Euclidean norm on space R^{nM}.

It can be derived from the above that

$$\sup_\nu\left\{\sum_{j=1}^{n}|\Pi_{\hat{K}}\hat{v}_j^\nu-\Pi_{\partial\hat{K}}\hat{v}_j^\nu|_{0,\partial\hat{K}}^2+\sum_{i,j=1}^{n}|\partial_i\widehat{\Pi_{K_\nu}v_j^\nu}+\partial_j\widehat{\Pi_{K_\nu}v_i^\nu}|_{0,\hat{K}}^2\right\}<\infty.$$

Hence

$$\lim_{\nu\to\infty}\sum_{i,j=1}^{n}|\widehat{E_{K_\nu}^{ij}(v^\nu)}|_{0,\hat{K}}^2=0. \quad (4.9)$$

By the fact that $\{\hat{\phi}(\hat{v}^\nu)\}$ and $\{B_{K_\nu}\}$ are bounded, there exist subsequences of $\{\hat{\phi}(\hat{v}^\nu)\}$ and $\{B_{K_\nu}\}$, also denoted by $\{\hat{\phi}(\hat{v}^\nu)\}$ and $\{B_{K_\nu}\}$, and a matrix B and $\hat{v}$ in $(H^1(\hat{K}))^n$, such that

$$\lim_{\nu\to\infty}\|\hat{\phi}(\hat{v}^\nu)-\hat{\phi}(\hat{v})\|=0,\quad \|\hat{\phi}(\hat{v})\|=1, \quad (4.10)$$

$$\lim_{\nu\to\infty}\|B_{K_\nu}-B\|=0,\quad |\det B|=1, \quad (4.11)$$

where $\|\cdot\|$ is a norm for matrices. From this we can get, $1\le j\le n$,

$$\Pi_{\hat{K}}\hat{v}_j^\nu\to\Pi_{\hat{K}}\hat{v}_j,\quad \Pi_{\partial\hat{K}}\hat{v}_j^\nu\to\Pi_{\partial\hat{K}}\hat{v}_j,\quad \text{as } \nu\to\infty.$$

Using transformation of coordinates, $x=F_{K_\nu}\hat{x}$, in equations (2.1), and the following equalities:

$$ds=|\det B_{K_\nu}|\,\|B_{K_\nu}^{-T}\hat{N}\|d\hat{s},\quad N=B_{K_\nu}^{-T}\hat{N}/\|B_{K_\nu}^{-T}\hat{N}\|, \quad (4.12)$$

we get, for $i, j=1, \cdots, n$, $\forall\hat{p}\in N_{\hat{K}}$,

$$2\int_{\hat{K}}\hat{p}\widehat{E_{K_\nu}^{ij}(v^\nu)}d\hat{x}=\int_{\partial\hat{K}}\hat{p}[(B_{K_\nu}^{-T}\hat{N})_j\Pi_{\partial\hat{K}}\hat{v}_i^\nu+(B_{K_\nu}^{-T}\hat{N})_i\Pi_{\partial\hat{K}}\hat{v}_j^\nu]d\hat{s}$$
$$-\sum_{l=1}^{n}\int_{\hat{K}}(B_{K_\nu}^{li}\partial_{\hat{x}_l}\hat{p}\Pi_{\hat{K}}\hat{v}_j^\nu+B_{K_\nu}^{lj}\partial_{\hat{x}_l}\hat{p}\Pi_{\hat{K}}\hat{v}_i^\nu)d\hat{x}, \quad (4.13)$$

where $B_{K_\nu}^{-T}=(B_{K_\nu}^{ij})_{n\times n}$. Letting $\nu\to\infty$ in the above equalities, we obtain for $i, j=1, \cdots, n$, $\forall\hat{p}\in N_{\hat{K}}$,

$$0=\int_{\partial\hat{K}}\hat{p}[(B^{-T}\hat{N})_j\Pi_{\partial\hat{K}}\hat{v}_i+(B^{-T}\hat{N})_i\Pi_{\partial\hat{K}}\hat{v}_j]d\hat{s}$$
$$-\sum_{l=1}^{n}\int_{\hat{K}}(B^{li}\partial_{\hat{x}_l}\hat{p}\Pi_{\hat{K}}\hat{v}_j+B^{lj}\partial_{\hat{x}_l}\hat{p}\Pi_{\hat{K}}\hat{v}_i)d\hat{x}, \quad (4.14)$$

where $B^{-T}=(B^{ij})_{n\times n}$. Define $F_K\hat{x}=B\hat{x}$ and $K=F_K\hat{K}$. Then by Lemma 1, it follows that

$$\hat{\phi}(\hat{v})\in\operatorname{Ker} Q_K. \tag{4.15}$$

Since $B_{K_\nu}\to B$, $Q_{K_\nu}\to Q_K$ as $\nu\to\infty$. So we can conclude that if $\hat{\phi}^\nu\in\operatorname{Ker} Q_{K_\nu}$ and $\hat{\phi}^\nu\to\hat{\phi}$ as $\nu\to\infty$, then $\hat{\phi}\in\operatorname{Ker} Q_K$.

Now we select a unit vector $\hat{\Phi}_1^\nu$ in $\operatorname{Ker} Q_{K_\nu}$ for each ν; then there exists a convergent subsequence of $\{\hat{\Phi}_1^\nu\}$, also denoted by $\{\hat{\Phi}_1^\nu\}$. Next we select a unit vector $\hat{\Phi}_2^\nu$ in $\operatorname{Ker} Q_{K_\nu}$ such that $\hat{\Phi}_1^\nu$, $\hat{\Phi}_2^\nu$ are orthogonal. By the same argument, there exists a convergent subsequence of $\{\hat{\Phi}_2^\nu\}$, also denoted by $\{\hat{\Phi}_2^\nu\}$.

Continuing with these steps, we can get $n(n+1)/2$ convergent sequences $\{\hat{\Phi}_m^\nu\}$, $m=1, 2, \cdots, n(n+1)/2$, such that they are an orthogonal basis of $\operatorname{Ker} Q_{K_\nu}$. Set

$$\hat{\Phi}_m=\lim_{\nu\to\infty}\hat{\Phi}_m^\nu,\quad m=1, 2, \cdots, n(n+1)/2.$$

Then $\hat{\Phi}_1$, $\hat{\Phi}_2$, $\cdots$, $\hat{\Phi}_{n(n+1)/2}$ are an orthogonal basis of $\operatorname{Ker} Q_K$. Thus if $\hat{\Phi}^\nu\in(\operatorname{Ker} Q_{K_\nu})^\perp$ and $\hat{\Phi}^\nu\to\hat{\Phi}$ then $\hat{\Phi}\in(\operatorname{Ker} Q_K)^\perp$. From (4.8) and (4.10), we get $\hat{\phi}(\hat{v})$ in $(\operatorname{Ker} Q_K)^\perp$, and $\hat{\phi}(\hat{v})=0$ by (4.15). On the other hand, $\|\hat{\phi}(\hat{v})\|=1$. They are contradictious. Hence inequality (4.6) holds and so does inequality (4.5).

Now we are in a position to prove inequalities (4.3) and (4.4). For each K in K_h, let $\tilde{x}=\theta_K x$, and $\tilde{K}=\{\tilde{x}\,|\,\tilde{x}=\theta_K x,\ \forall x\in K\}$, where

$$\theta_K^n=(\text{volume of }\hat{K}/\text{volume of }K),\ C_1h^{-1}\leqslant\theta_K\leqslant C_2h^{-1}. \tag{4.16}$$

Then $\tilde{K}$ is in J_{C,η_1} by condition K3. Denote $\tilde{v}_i(\tilde{x})=v_i(\theta_K^{-1}\tilde{x})$. By inequalities (4.5) we have

$$\sum_{i,j=1}^{n}|\partial_i\Pi_K v_j+\partial_j\Pi_K v_i|_{0,K}=\theta_K^{1-n/2}\sum_{i,j=1}^{n}|\partial_{\tilde{x}_i}\Pi_{\tilde{K}}\tilde{v}_j+\partial_{\tilde{x}_j}\Pi_{\tilde{K}}\tilde{v}_i|_{0,\tilde{K}}$$

$$\leqslant C(C_2\eta_1)\theta_K^{1-n/2}\sum_{i,j=1}^{n}|E_{\tilde{K}}^{ij}(\tilde{v})|_{0,\tilde{K}}\leqslant C\sum_{i,j=1}^{n}|E_K^{ij}(v)|_{0,K},$$

and

$$\sum_{j=1}^{n}|\Pi_K v_j-\Pi_{\partial K}v_j|_{0,\partial K}=\theta_K^{(1-n)/2}\sum_{j=1}^{n}|\Pi_{\tilde{K}}\tilde{v}_j-\Pi_{\partial\tilde{K}}\tilde{v}_j|_{0,\partial\tilde{K}}$$

$$\leqslant C\theta_K^{(1-n)/2}\sum_{i,j=1}^{n}|E_{\tilde{K}}^{ij}(\tilde{v})|_{0,\tilde{K}}\leqslant C\theta_K^{-1/2}\sum_{i,j=1}^{n}|E_K^{ij}(v)|_{0,K}$$

$$\leqslant Ch^{1/2}\sum_{i,j=1}^{n}|E_K^{ij}(v)|_{0,K}.$$

These are inequalities (4.3) and (4.4). Lemma 2 is proved.

Lemma 3. *There exists a constant $C>0$ independent of F, K and h such that*

$$\sum_{j=1}^{n}|\Pi_{\partial K}v_j-\Pi_{\partial K'}v_j|_{0,F}\leqslant Ch^{1/2}\sum_{i,j=1}^{n}\{|E_K^{ij}(v)|_{0,K}+|E_{K'}^{ij}(v)|_{0,K'}\} \tag{4.17}$$

holds for all v in $(H^t(\Omega))^n$ when $F=K\cap K'$ is an $(n-1)$-dimensional face of both K and K' in K_h and

$$\sum_{j=1}^{n}|\Pi_{\partial K}v_j|_{0,F}\leqslant Ch^{1/2}\sum_{i,j=1}^{n}|E_K^{ij}(v)|_{0,K} \tag{4.18}$$

holds for all v in $(H^t(\Omega)\cap H_0^1(\Omega))^n$ when $F=K\cap\partial\Omega$ is an $(n-1)$-dimensional face of K in K_h.

Proof. Let K and K' be any two n-simplexes (or n-parallelotopes) with a common $(n-1)$-dimensional face F. We define a space $H(K, K')=\{(v_j)\,|\,v_j\in L^2(K\cup K'),\ v_j|_K\in H^t(K),\ v_j|_{K'}\in H^t(K')$, and v_j are continuous at the vertices of $F\}$.

First, we have the conclusion that for such two elements there exists a constant $C(K, K')$, such that

$$\sum_{j=1}^{n}|\Pi_{2K}v_j-\Pi_{2K'}v_j|_{0,F}\leqslant C(K,\ K')\sum_{i,j=1}^{n}\{|E_K^{ij}(v)|_{0,K}+|E_{K'}^{ij}(v)|_{0,K'}\} \tag{4.19}$$

holds for all v in $H(K, K')$. In fact, if $E_K^{ij}(v)=0$ and $E_{K'}^{ij}(v)=0$, then $\Pi_{2K}v_j$ and $\Pi_{2K'}v_j$ are polynomials of degree not greater than one by Lemma 2. On the other hand, $\Pi_{2K}v_j|_F=\Pi_{2K'}v_j|_F$ because the values of $\Pi_{2K}v_j|_F$ and $\Pi_{2K'}v_j|_F$ at the vertices of F are the same. It follows that inequality (4.19) holds.

Secondly, we prove that there exists a constant $C(\xi_1,\ \xi_2)$ independent of K and K' and v such that

$$\sum_{j=1}^{n}|\Pi_{2K}v_j-\Pi_{2K'}v_j|_{0,F}\leqslant C(\xi_1,\ \xi_2)\sum_{i,j=1}^{n}\{|E_K^{ij}(v)|_{0,K}+|E_{K'}^{ij}(v)|_{0,K'}\} \tag{4.20}$$

holds for v in $H(K, K')$ and all K and K' with the property that K has the same volume as $\hat{K}$ does and $h_K\leqslant\xi_1$, $h_{K'}\leqslant\xi_1$, $\rho_{K'}\geqslant\xi_2$.

In order to get inequality (4.20), we assume that it does not hold. Then for $\nu=1, 2, \cdots$, there exist K_ν, K'_ν and v^ν in $H(K_\nu, K'_\nu)$, such that

$$\sum_{j=1}^{n}|\Pi_{2K_\nu}v_j^\nu-\Pi_{2K'_\nu}v_j^\nu|_{0,F_\nu}>\nu\sum_{i,j=1}^{n}\{|E_{K_\nu}^{ij}(v^\nu)|_{0,K_\nu}+|E_{K'_\nu}^{ij}(v^\nu)|_{0,K'_\nu}\}, \tag{4.21}$$

where $F_\nu=K_\nu\cap K'_\nu$ is an $(n-1)$-dimensional face of both K_ν and K'_ν.

If $v\in(P_1(K_\nu\cup K'_\nu))^n$ is a rigid displacement, then $\Pi_{2K_\nu}v_j|_{F_\nu}=\Pi_{2K'_\nu}v_j|_{F_\nu}$ and $E_{K_\nu}^{ij}(v)=0$ and $E_{K'_\nu}^{ij}(v)=0$ for $1\leqslant i,\ j\leqslant n$. So we can assume that

$$\hat{\phi}(\widehat{v^\nu|_{K_\nu}})\in(\mathrm{Ker}\,Q_{K_\nu})^\perp,\quad \nu=1,\ 2,\ \cdots, \tag{4.22}$$

$$\|\hat{\phi}(\widehat{v^\nu|_{K_\nu}})\|^2+\|\hat{\phi}(\widehat{v^\nu|_{K'}})\|^2=1,\quad \nu=1,\ 2,\ \cdots. \tag{4.23}$$

From inequality (4.21), we derive that inequality

$$\sum_{j=1}^{n}|\Pi_{2\hat{K}}(\widehat{v_j^\nu|_{K_\nu}}-\widehat{v_j^\nu|_{K'_\nu}})|_{0,\hat{F}}>C\nu\sum_{i,j=1}^{n}\{|E_{\hat{K}_\nu}^{ij}(\widehat{v^\nu|_{K_\nu}})|_{0,\hat{K}}+|E_{\hat{K}'_\nu}^{ij}(\widehat{v^\nu|_{K'_\nu}})|_{0,\hat{K}}\}$$

holds, where $\hat{F}$ is a fixed $(n-1)$-dimensional face of $\hat{K}$. Hence

$$\lim_{\nu\to\infty}\sum_{i,j=1}^{n}\{|E_{\hat{K}_\nu}^{ij}(\widehat{v^\nu|_{K_\nu}})|_{0,\hat{K}}+|E_{\hat{K}'_\nu}^{ij}(\widehat{v^\nu|_{K'_\nu}})|_{0,\hat{K}}\}=0. \tag{4.24}$$

Similarly to the proof of Lemma 2, there exist subsequences of the sequences $\{\hat{\phi}(\widehat{v^\nu|_{K_\nu}})\}$, $\{\hat{\phi}(\widehat{v^\nu|_{K'_\nu}})\}$, $\{B_{K_\nu}\}$ and $\{B_{K'_\nu}\}$, denoted likewise, and $\hat{v}$, $\hat{v}'$ in $(H^t(\hat{K}))^n$, and nonsingular matrices B and B', such that

$$\lim_{\nu\to\infty}\|\hat{\phi}(\widehat{v^\nu|_{K_\nu}})-\hat{\phi}(\hat{v})\|=\lim_{\nu\to\infty}\|\hat{\phi}(\widehat{v^\nu|_{K'_\nu}})-\hat{\phi}(\hat{v}')\|=0,\quad \|\hat{\phi}(\hat{v})\|^2+\|\hat{\phi}(\hat{v}')\|^2=1,$$

$$\lim_{\nu\to\infty}\|B_{K_\nu}-B\|=\lim_{\nu\to\infty}\|B_{K'_\nu}-B'\|=0. \tag{4.25}$$

Now define $F_K\hat{x}=B\hat{x}$ and $F_{K'}\hat{x}=B'\hat{x}$, and set $K=F_K\hat{K}$ and $K'=F_{K'}\hat{K}$. By the same argument as in the proof of Lemma 2, we get $\hat{\phi}(\hat{v})\in\operatorname{Ker}Q_K$ and $\hat{\phi}(\hat{v}')\in\operatorname{Ker}Q_{K'}$ and $\hat{\phi}(\hat{v})=0$. Since $\hat{\phi}(\hat{v}')$ is in Ker $Q_{K'}$, $\Pi_{\partial\hat{K}}\hat{v}'_j|_{\hat{F}}$ are polynomials of degree at most one. On the other hand, $\Pi_{\partial\hat{K}}\hat{v}'_j|_{\hat{F}}$ and $\Pi_{\partial\hat{K}}\hat{v}_j|_{\hat{F}}$ have the same values at the vertices of $\hat{F}$, and $\Pi_{\partial\hat{K}}\hat{v}_j|_{\hat{F}}=0$; so $\Pi_{\partial\hat{K}}\hat{v}'_j|_{\hat{F}}=0$. This means that $\hat{\phi}(\hat{v}')=0$. It is contradictious to $\|\hat{\phi}(\hat{v})\|^2+\|\hat{\phi}(\hat{v}')\|^2=1$. Thus inequality (4.20) holds.

Finally, for each two elements K and K' in K_h, we can prove that inequality (4.17) holds in the same way as in the proof of Lemma 2. Similarly, we can get inequality (4.18).

Lemma 4. *$|\cdot|_E$ is a norm on U_h and problem* (1.7) *has a unique solution.*

Proof. It suffices to show that if $\Lambda_h\in U_h$ and $|\Lambda_h|_E=0$ then $\Lambda_h=0$. Let $\Lambda_h=\Pi_h v$, $v\in(H^t(\Omega)\cap H_0^1(\Omega))^n$. From $E_h^{ij}v=0$, $1\leqslant i$, $j\leqslant n$ and Lemmas 2 and 3, we get

$$\partial_i\Pi_K v_j+\partial_j\Pi_K v_i=0,\ \Pi_{\partial K}v_j-\Pi_K v_j|_{\partial K}=0,$$
$$1\leqslant i,\ j\leqslant n,\ K\in K_h.$$

By condition 4) of the theorem and Lemma 3, if $F=K\cap K'$ is an $(n-1)$-dimensional face of both K and K', then $\Pi_K v_j|_F=\Pi_{K'}v_j|_F$, $1\leqslant j\leqslant n$, and if $F=K\cap\partial\Omega$ is an $(n-1)$-dimensional face, then $\Pi_K v_j|_F=0$ and so $\Pi_K v_j=0$ on K, $1\leqslant j\leqslant n$. From this we can conclude that $\Pi_K v_j=0$ for $j=1,\cdots,n$ and K in K_h. This means that $\Pi_h v=0$.

It is obvious that $a(\cdot,\cdot)$ is uniformly U_h-elliptic with respect to norm $|\cdot|_E$; so the following inequality[5]

$$|Tu-\Lambda_h^*|_E\leqslant C\left\{\min_{\Lambda_h\in U_h}|Tu-\Lambda_h|_E+\sup_{0\neq\Lambda_h\in U_h}\frac{|a(Tu,\Lambda_h)-f(\Lambda_h)|}{|\Lambda_h|_E}\right\}\tag{4.26}$$

is uniformly true for all h, where C is a constant independent of u and h.

Lemma 5. *There exists a constant C independent of u and h, such that when u is in $(H^r(\Omega)\cap H_0^1(\Omega))^n$, the inequality*

$$\min_{\Lambda_h\in U_h}|Tu-\Lambda_h|_E\leqslant C\{h^{r_1}|u|_{r_1+1,\Omega}+h^{r_1+1}|u|_{r_1+2,\Omega}\}\tag{4.27}$$

is true for all h.

Proof. It is sufficient to show that there exists a constant C independent of u, h and K such that

$$|e_{ij}(u)-E_K^{ij}(u)|_{0,K}\leqslant C\{h^{r_1}|u|_{r_1+1,K}+h^{r_1+1}|u|_{r_1+2,K}\},\quad 1\leqslant i,\ j\leqslant n\tag{4.28}$$

hold for each K in K_h and each h. And inequalities (4.28) can be obtained in a way similar to that used in [1].

Lemma 6. *There exists a constant C independent of u and h, such that*

$$|a(Tu,\Lambda_h)-f(\Lambda_h)|\leqslant C\{h^{r_1+1}|u|_{r_1+2,\Omega}+h^{r_1+1}|u|_{r_1+2,\Omega}\}|\Lambda_h|_E\tag{4.29}$$

hold for all $\Lambda_h\in U_h$ and h.

Proof. Let $\Lambda_h=\Pi_h v$ and v in $(H^t(\Omega)\cap H_0^1(\Omega))^n$. Noticing u is the solution of problem (1.2), we have

$$
\begin{aligned}
a(Tu, \Lambda_h) - f(\Lambda_h) &= \sum_K \int_K \sum_{i,j=1}^n (\sigma_{ij}(u) E_K^{ij}(v) + \partial_j \sigma_{ij}(u) \Pi_K v_i)\, dx \\
&= \sum_K \sum_{i,j=1}^n \Big\{ \int_K \sigma_{ij}(u) [E_K^{ij}(v) - (\partial_i \Pi_K v_j + \partial_j \Pi_K v_i)/2]\, dx \\
&\quad + \int_{\partial K} \sigma_{ij}(u) (\Pi_K v_i N_j + \Pi_K v_j N_i)/2\, ds \Big\} \\
&= \sum_K \sum_{i,j=1}^n \Big\{ \int_K \sigma_{ij}(u) [E_K^{ij}(v) - (\partial_i \Pi_K v_j + \partial_j \Pi_K v_i)/2]\, dx \\
&\quad + \int_{\partial K} \sigma_{ij}(u) [(\Pi_K v_i - \Pi_{\partial K} v_i) N_j + (\Pi_K v_j - \Pi_{\partial K} v_j) N_i]/2\, ds \Big\} \\
&\quad + \sum_K \sum_{i,j=1}^n \int_{\partial K} \sigma_{ij}(u) \Pi_{\partial K} v_i N_j\, ds.
\end{aligned}
$$

Let $P_K^{r_2}$: $L^2(K) \to P_{r_2}(K)$ be the orthogonal projection operator in $L^2(K)$ sense and $P_F^{r_3}$: $L^2(F) \to P_{r_3}(F)$ the orthogonal projection operator in $L^2(F)$ sense when F is an $(n-1)$-dimensional face of K. By conditions 2 and 3 of the theorem and equations (2.1), we get

$$
\begin{aligned}
a(Tu, \Lambda_h) - f(\Lambda_h) &= \sum_K \sum_{i,j=1}^n \Big\{ \int_K [\sigma_{ij}(u) - P_K^{r_2} \sigma_{ij}(u)] [E_K^{ij}(v) - (\partial_i \Pi_K v_j + \partial_j \Pi_K v_i)/2]\, dx \\
&\quad + \int_{\partial K} [\sigma_{ij}(u) - P_K^{r_2} \sigma_{ij}(u)] [(\Pi_K v_j - \Pi_{\partial K} v_j) N_i \\
&\quad + (\Pi_K v_i - \Pi_{\partial K} v_i) N_j]/2\, ds \Big\} \\
&\quad + \sum_K \sum_{i,j=1}^n \sum_{F \subset \partial K} \int_F [\sigma_{ij}(u) - P_F^{r_3} \sigma_{ij}(u)] \Pi_{\partial K} v_i N_j\, ds.
\end{aligned}
$$

Using the interpolation theory[1,5] and Lemmas 2 and 3, we can get

$$
\begin{aligned}
|a(Tu, \Lambda_h) - f(\Lambda_h)| &\le C \sum_K \Big(\sum_{i,j=1}^n |E_K^{ij}(v)|_{0,K}^2 \Big)^{1/2} \Big(\sum_{l=2}^3 h^{r_l+1} |u|_{r_l+2,K} \Big) \\
&\le C \sum_{l=2}^3 h^{r_l+1} |u|_{r_l+2,\Omega} \|\Pi_h v\|_H.
\end{aligned}
$$

Hence Lemma 6 is true.

We can immediately get the conclusions of the theorem from Lemmas 4, 5 and 6 and inequalities (4.26).

Proof of the corollary. Using the classical duality argument, we have

$$
\Big(\sum_{j=1}^n \|u_j - (u_h^*)_j\|_{0,\Omega}^2 \Big)^{1/2} = \sup_{g \in (L^2(\Omega))^n} \Big| \sum_{j=1}^n (u_j - (u_h^*)_j, g_j) \Big| / \|g\|_{0,\Omega}, \tag{4.30}
$$

where $(\cdot, \cdot)$ is $L^2(\Omega)$ inner. For each g in $(L^2(\Omega))^n$, denote by

$$
\varphi_g \in (H^2(\Omega) \cap H_0^1(\Omega))^n
$$

and $\varphi_h^* \in U_h$ the solutions of the following problems respectively:

$$
a(Tv, T\varphi_g) = g(Tv), \quad \forall v \in (H_0^1(\Omega))^n, \tag{4.31}
$$

$$
a(\Lambda_h, \varphi_h^*) = g(\Lambda_h), \quad \forall \Lambda_h \in U_h, \tag{4.32}
$$

where $g(\Lambda) = \sum_{j=1}^n (V_j, g_j)$ for $\Lambda \in H$. Then

$$\sum_{j=1}^{n}(u_j-(u_h^*)_j,\ g_j)=a(Tu,\ T\varphi_g)-g(\Lambda_h^*)$$
$$=a(Tu-\Lambda_h^*,\ T\varphi_g-\varphi_h^*)+a(Tu,\ \varphi_h^*)$$
$$-f(\varphi_h^*)+a(\Lambda_h^*,\ T\varphi_g)-g(\Lambda_h^*),$$

$$\left|\sum_{j=1}^{n}(u_j-(u_h^*)_j,g_j)\right|\leqslant C|Tu-\Lambda_h^*|_B|T\varphi_g-\varphi_h^*|_B$$
$$+|a(Tu,\ \varphi_h^*)-f(\varphi_h^*)|+|a(\Lambda_h^*,\ T\varphi_g)-g(\Lambda_h^*)|. \tag{4.33}$$

Let $\Lambda_h^*=\Pi_h v$, $\varphi_h^*=\Pi_h\varphi$, for v, φ in $(H^t(\Omega)\cap H_0^1(\Omega))^n$. We have

$$a(\Lambda_h^*,\ T\varphi_g)-g(\Lambda_h^*)=\sum_K\sum_{i,j=1}^{n}\int_K[\sigma_{ij}(\varphi_g)E_K^{ij}(v)+\partial_j\sigma_{ij}(\varphi_g)\Pi_K v_i]dx$$
$$=\sum_K\sum_{i,j=1}^{n}\Big\{\int_K[\sigma_{ij}(\varphi_g)-P_K^0\sigma_{ij}(\varphi_g)]$$
$$\times[E_K^{ij}(v)-(\partial_j\Pi_K v_i+\partial_i\Pi_K v_j)/2]dx$$
$$+\int_{\partial K}[\sigma_{ij}(\varphi_g)-P_K^0\sigma_{ij}(\varphi_g)][(\Pi_K v_j-\Pi_{\partial K}v_j)N_i]ds\Big\}$$
$$+\sum_K\sum_{i,j=1}^{n}\sum_{F\subset\partial K}\int_F[\sigma_{ij}(\varphi_g)-P_F^0\sigma_{ij}(\varphi_g)]\Pi_{\partial K}v_iN_j\,ds.$$

Hence

$$|a(\Lambda_h^*,\ T\varphi_g)-g(\Lambda_h^*)|\leqslant Ch|\varphi_g|_{2,\Omega}\Big(\sum_K\sum_{i,j=1}^{n}|E_K^{ij}(v)-(\partial_j\Pi_K v_i+\partial_i\Pi_K v_j)/2|_{0,K}^2\Big)^{1/2}$$
$$+Ch^{1/2}|\varphi_g|_{2,\Omega}\Big\{\Big(\sum_K\sum_{i,j=1}^{n}|\Pi_K v_j-\Pi_{\partial K}v_j|_{0,K}^2\Big)^{1/2}$$
$$+\sum_{j=1}^{n}\Big(\sum_{F=K\cap K'}|\Pi_{\partial K}v_j-\Pi_{\partial K'}v_j|_{0,F}^2+\sum_{F=K\cap\partial\Omega}|\Pi_{\partial K}v_j|_{0,F}^2\Big)^{1/2}\Big\}.$$

Using Lemmas 2 and 3 and inequalities (4.28), we get

$$|E_K^{ij}(v)-(\partial_i\Pi_K v_j+\partial_j\Pi_K v_i)/2|_{0,K}$$
$$\leqslant|E_K^{ij}(v)-\varepsilon_{ij}(u)|_{0,K}+|\varepsilon_{ij}(u)-(\partial_j\Pi_K u_i+\partial_i\Pi_K v_j)/2|_{0,K}$$
$$+|(\partial_j\Pi_K(u-v)_i+\partial_i\Pi_K(u-v)_j)/2|_{0,K}$$
$$\leqslant|E_K^{ij}(v)-\varepsilon_{ij}(u)|_{0,K}+C\Big\{h^{r_1}|u|_{r_1+1,K}+\sum_{l,m=1}^{n}|E_K^{lm}(u)-E_K^{lm}(v)|_{0,K}\Big\}$$
$$\leqslant C\Big\{h^{r_1}|u|_{r_1+2,K}+h^{r_1+1}|u|_{r_1+2,K}+\sum_{l,m=1}^{n}|E_K^{lm}(v)-\varepsilon_{lm}(u)|_{0,K}\Big\},$$

$$|\Pi_K v_j-\Pi_{\partial K}v_j|_{0,\partial K}\leqslant|\Pi_K(v-u)_j-\Pi_{\partial K}(v-u)_j|_{0,\partial K}$$
$$+|u_j-\Pi_K u_j|_{0,\partial K}+|u_j-\Pi_{\partial K}u_j|_{0,\partial K}$$
$$\leqslant C\Big\{h^{1/2}\sum_{l,m=1}^{n}|E_K^{lm}(v-u)|_{0,K}+h^{r_1+1/2}|u|_{r_1+1,K}\Big\}$$
$$\leqslant C\Big\{h^{r_1+1/2}|u|_{r_1+1,K}+h^{r_1+3/2}|u|_{r_1+2,K}$$
$$+h^{1/2}\sum_{l,m=1}^{n}|E_K^{lm}(v)-\varepsilon_{lm}(u)|_{0,K}\Big\}.$$

When $F=K\cap K'$,

$$|\Pi_{\partial K}v_j-\Pi_{\partial K'}v_j|_{0,F}\leqslant|\Pi_{\partial K}(v-u)_j-\Pi_{\partial K'}(v-u)_j|_{0,F}$$
$$+|\Pi_{\partial K}u_j-u_j|_{0,F}+|\Pi_{\partial K'}u_j-u_j|_{0,F}$$
$$\leqslant C\left\{h^{r_1+1/2}|u|_{r_1+1,K\cup K'}+h^{r_2+3/2}|u|_{r_2+2,K\cup K'}\right.$$
$$\left.+h^{1/2}\sum_{l,m=1}^{n}|E_h^{lm}v-s_{lm}(u)|_{0,K\cup K'}\right\},$$

and when $F=K\cap\partial\Omega$,

$$|\Pi_{\partial K}v_j|_{0,F}\leqslant C\left\{h^{r_1+1/2}|u|_{r_1+1,K}+h^{r_2+3/2}|u|_{r_2+2,K}+h^{1/2}\sum_{l,m=1}^{n}|E_K^{lm}(v)-s_{lm}(u)|_{0,K}\right\}.$$

It follows that the following inequalities

$$|a(\Lambda_h^*,T\varphi_g)-g(\Lambda_h^*)|\leqslant Ch|\varphi_g|_{2,\Omega}\{h^{r_1}|u|_{r_1+1,\Omega}+h^{r_2+1}|u|_{r_2+2,\Omega}+|Tu-\Lambda_h^*|_B\}\tag{4.34}$$

hold. Under the conditions of the corollary, the integers r_1, r_2 and r_3 in the theorem can be 1, 0 and 0 respectively. Thus in a similar way, we get

$$|a(Tu,\varphi_h^*)-f(\varphi_h^*)|\leqslant C(h^{r_2+1}|u|_{r_2+2,\Omega}+h^{r_3+1}|u|_{r_3+2,\Omega})(|T\varphi_g-\varphi_h^*|_B+h|\varphi_g|_{2,\Omega}).\tag{4.35}$$

Notice

$$|T\varphi_g-\varphi_h^*|_B\leqslant Ch|\varphi_g|_{2,\Omega},$$
$$|Tu-\Lambda_h^*|_B\leqslant C\{h^{r_1}|u|_{r_1+1,\Omega}+h^{r_2+1}|u|_{r_2+2,\Omega}+h^{r_3+1}|u|_{r_3+2,\Omega}\}.$$

We can obtain

$$\left|\sum_{j=1}^{n}(u_j-(u_h^*)_j,g_j)\right|\leqslant C\left\{h^{r_1+1}|u|_{r_1+1,\Omega}+\sum_{i=2}^{3}h^{r_i+2}|u|_{r_i+2,\Omega}\right\}|\varphi_g|_{2,\Omega}.\tag{4.36}$$

Finally, we have

$$|\varphi_g|_{2,\Omega}\leqslant C\|g\|_{0,\Omega}.\tag{4.37}$$

So the conclusion of the corollary is true.

§ 5. Curved Elements for Second-Order Boundary Value Problems

Using the method of preceding sections, we can easily get similar results for second-order boundary value problems, and these results can be generalized to the case of curved elements. For convenience we take the Dirichlet problem of Poisson equation as an example to describe these results.

Again, let Ω be a bounded domain in R^n with Lipschitz-continuous boundary $\partial\Omega$. Set $L^{1,2}(\Omega)=\{u=(u^0,u^1,\cdots,u^n)|u^i\in L^2(\Omega),\ i=0,1,\cdots,n\}$. For each v in $H^1(\Omega)$, let $\tilde{T}v=(v,\partial_1v,\cdots,\partial_nv)$. Then $\tilde{T}H^1(\Omega)$ is a subspace of $L^{1,2}(\Omega)$. To describe the problem, we define a bilinear form $\tilde{a}(\cdot,\cdot)$ and a linear form $\tilde{f}(\cdot)$ on space $L^{1,2}(\Omega)$ as follows:

$$\tilde{a}(u,v)=\int_\Omega\sum_{i=1}^{n}u^iv^i\,dx,\quad\forall u,v\text{ in }L^{1,2}(\Omega),\tag{5.1}$$

$$\tilde{f}(u)=\int_\Omega\tilde{f}u^0\,dx,\quad\forall u\text{ in }L^{1,2}(\Omega),\tag{5.2}$$

where $\tilde{f}$ is in $L^2(\Omega)$.

Let U_h be a family of finite dimensional subspaces in $L^{1,2}(\Omega)$ for parameter h with $h>0$ and $h\to 0$. Consider the following variational problems:

$$u_0\in H_0^1(\Omega),\quad \tilde{a}(\tilde{T}u_0,\tilde{T}v)=\tilde{f}(\tilde{T}v),\quad \forall v\in H_0^1(\Omega),\tag{5.3}$$

$$u_h\in U_h,\quad \tilde{a}(u_h,v_h)=\tilde{f}(v_h),\quad \forall v_h\in U_h.\tag{5.4}$$

We want to use the solution u_h of problem (5.4) to approximate that of problem (5.3). In this section we will show how to construct U_h with isoparameter quasi-conforming element method and give the convergence conditions and error estimates of solution u_h.

Let $\hat{K}$ be a fixed n-simplex or n-rectangle. For each h, let K_h be a subdivision of domain Ω with the following properties: i) $\bigcup_{K\in K_h} K=\bar{\Omega}$, ii) for each K in K_h, there exists a sufficiently smooth invertible map F_K: $\hat{K}\to K$, such that $K=F_K\hat{K}$ and the Jacobian J_K of map F_K does not vanish at every $\hat{x}$ in $\hat{K}$, and the following inequalities are true:

$$c_1h^n\leqslant |J_K|_{0,\infty,\hat{K}}\leqslant c_2h^n,\tag{5.5}$$

$$|F_K^{-1}|_{1,\infty,K}\leqslant c_3h^{-1},\tag{5.6}$$

$$|F_K|_{l,\infty,\hat{K}}\leqslant c_4h^l,\quad l=1,\cdots,\mathscr{L},\tag{5.7}$$

where c_i are positive constants independent of K and h, $\mathscr{L}$ is a sufficiently large integer, iii) let $\hat{F}$ be a face of $\hat{K}$; we call $F=F_K\hat{F}$ a face of element K. Then for every two different elements K and K' in K_h, $K\cap K'$ is either a face of both K and K' or an empty set, and when $K\cap K'$ is not empty, we assume that $K\cap K'=F_K\hat{F}=F_{K'}\hat{F}$ and $F_K|_{\hat{F}}=F_{K'}|_{\hat{F}}$.

Now we consider the approximation of functions on K. Let t be a positive integer. First, we give two interpolation operators $\Pi_{\hat{K}}^0$: $H^t(\hat{K})\to L^2(\hat{K})$ and $\Pi_{\partial\hat{K}}$: $H^t(\hat{K})\to L^2(\partial\hat{K})$ and n finite dimensional spaces $N_{\hat{K}}^i$ consisting of polynomials on $\hat{K}$, $i=1,\cdots,n$. Then define linear operators $\Pi_{\hat{K}}^i$: $H^t(\hat{K})\to N_{\hat{K}}^i$ for $i=1,\cdots,n$, such that, for each $\hat{u}$ in $H^t(\hat{K})$, $\Pi_{\hat{K}}^i\hat{u}$ is determined by the following equations:

$$\int_{\hat{K}}\hat{p}\Pi_{\hat{K}}^i\hat{u}\,d\hat{x}=\int_{\partial\hat{K}}\hat{p}\Pi_{\partial\hat{K}}\hat{u}\hat{N}_i\,d\hat{s}-\int_{\hat{K}}\partial_i\hat{p}\Pi_{\hat{K}}^0\hat{u}\,d\hat{x},\quad \forall\hat{p}\in N_{\hat{K}}^i,\ 1\leqslant i\leqslant n.\tag{5.8}$$

Thus for $\hat{u}$ in $H^t(\hat{K})$, we use $\Pi_{\hat{K}}^i\hat{u}$ to approximate $\partial_i\hat{u}$ $(1\leqslant i\leqslant n)$ respectively (see [1, 2]).

Secondly, we define, for each u in $H^t(K)$,

$$\hat{u}(\hat{x})=u(F_K\hat{x}),\quad \forall\hat{x}\in\hat{K},\tag{5.9}$$

$$(\Pi_K^0u)(x)=(\Pi_{\hat{K}}^0\hat{u})(F_K^{-1}x),\quad \forall x\in K,\tag{5.10}$$

$$(\Pi_{\partial K}u)(x)=(\Pi_{\partial\hat{K}}\hat{u})(F_K^{-1}x),\quad \forall x\in\partial K,\tag{5.11}$$

$$((\Pi_K^1u)(x),\cdots,(\Pi_K^nu)(x))^T=(J_K^{-T})((\Pi_{\hat{K}}^1\hat{u})(F_K^{-1}x),\cdots,(\Pi_{\hat{K}}^n\hat{u})(F_K^{-1}x))^T,\quad \forall x\in K,\tag{5.12}$$

where (J_K^{-T}) is the Jacobi matrix of F_K^{-1}. Thus we obtain linear operators Π_K^0, $\Pi_{\partial K}$, $\Pi_K^1,\cdots,\Pi_K^n$ on $H^t(K)$.

Now we are in a position to construct space U_h. Define a linear operator Π_h: $H^t(\Omega)\to L^{1,2}(\Omega)$ as follows: for each u in $H^t(\Omega)$, Π_hu is determined by

$$(\Pi_hu)^i|_K=\Pi_K^i(u|_K),\quad K\in K_h,\ 0\leqslant i\leqslant n.\tag{5.13}$$

Set $U_h=\Pi_h(H^t(\Omega)\cap H_0^1(\Omega))$ and this is what we want.

In order to discuss the convergence conditions, we need more about interpolation operators $\Pi^0_{\hat{K}}$ and $\Pi_{\partial\hat{K}}$. Let $\hat{G}_1, \cdots, \hat{G}_M$ be a group of polynomials defined on $\hat{K}$ and $\hat{g}_1, \cdots, \hat{g}_M$ a group of functions in $L^2(\partial\hat{K})$, and let $\hat{\phi}_1, \cdots, \hat{\phi}_M$ be a group of linearly independent functionals on $H^t(\hat{K})$, which are called the parameters of $\Pi^0_{\hat{K}}$ and $\Pi_{\partial\hat{K}}$. Assume that for each $\hat{u}$ in $H^t(\hat{K})$, we have

$$\Pi^0_{\hat{K}}\hat{u}=\sum_{j=1}^{M}\hat{\phi}_j(\hat{u})\hat{G}_j, \quad \Pi_{\partial\hat{K}}\hat{u}=\sum_{j=1}^{M}\hat{\phi}_j(\hat{u})\hat{g}_j. \tag{5.14}$$

Then for each $\hat{u}$ in $H^t(\hat{K})$, denote

$$\hat{\phi}(\hat{u})=(\hat{\phi}_1(\hat{u}), \cdots, \hat{\phi}_M(\hat{u})).$$

Let Li be the dimension of $N^i_{\hat{K}}$, and choose a basis of $N^i_{\hat{K}}$. Denote the coordinate vector of $\Pi^i_{\hat{K}}\hat{u}$ with respect to this basis by $\beta_i(\hat{u})$, and define

$$\beta(\hat{u})=(\beta_1^T(\hat{u}), \cdots, \beta_n^T(\hat{u}))^T, \quad L=\sum_{i=1}^{n} Li.$$

Then equations (5.8) become

$$A_{\hat{K}}\beta(\hat{u})=Q_{\hat{K}}\hat{\phi}(\hat{u}), \tag{5.15}$$

where $A_{\hat{K}}$ is a nonsingular $L\times L$ matrix and $Q_{\hat{K}}$ an $L\times M$ matrix.

We define a semi-norm $|\cdot|_1$ an $L^{1,2}(\Omega)$ as follows:

$$|u|_1^2=\int_\Omega \sum_{i=1}^{n}|u^i|^2dx, \quad \forall u\in L^{1,2}(\Omega). \tag{5.16}$$

In a way as used in preceding sections, we can prove the following theorem.

Theorem. *Assume that the following four statements are true:*

1) *the rank of matrix $Q_{\hat{K}}$ is $M-1$;*

2) *there exists an integer $r_1\geq 1$, such that, for $\hat{p}$ in $P_{r_1}(\hat{K})$, $\Pi^0_{\hat{K}}\hat{p}=\hat{p}$ and $\Pi_{\partial\hat{K}}\hat{p}=\hat{p}|_{\partial\hat{K}}$, and $\Pi^0_{\hat{K}}$: $H^{r_1+1}(\hat{K})\to L^2(\hat{K})$ and $\Pi_{\partial\hat{K}}$: $H^{r_1}(\hat{K})\to L^2(\partial\hat{K})$ are continuous, and $N^i_{\hat{K}}\supset P_{r_1-1}(\hat{K})$;*

3) *there exists a nonnegative integer r_2, such that, for each $\hat{u}$ in $H^t(\hat{K})$, each $(n-1)$-dimensional face $\hat{F}$ of $\hat{K}$ and each $\hat{p}$ in $P_{r_2}(\hat{F})$, the integration $\int_{\hat{F}}\hat{p}\Pi_{\partial\hat{K}}\hat{u}d\hat{s}$ can be determined by the parameters of $\Pi_{\partial\hat{K}}\hat{u}$ at the interpolation points which lie on $\hat{F}$ and $\hat{F}$;*

4) *all the parameters of $\Pi_{\partial\hat{K}}$ are values of function and tangent derivative, and $\Pi^0_{\hat{K}}$ has at least one interpolation point of function on each $(n-1)$-dimensional face $\hat{F}$, and $\Pi^0_{\hat{K}}$ and $\Pi_{\partial\hat{K}}$ belong to some affine family. Then the following two conclusions are true:*

I. *the problem* (5.4) *has a unique solution u_h and $\lim_{h\to 0}|\tilde{T}u_0-u_h|_1=0$, and the error estimate*

$$|\tilde{T}u_0-u_h|_1\leq C\{h^{r_1}\|u_0\|_{r_1+1,\Omega}+h^{r_2+1}\|u_0\|_{r_2+2,\Omega}\} \tag{5.17}$$

is true when u_0 is in $H^r(\Omega)\cap H^1_0(\Omega)$, where C is a constant independent of h and u_0 and $r=\max\{r_1+1, r_2+2\}$;

II. *if Ω is convex and $\Pi^0_{\hat{K}}$: $H^2(\hat{K})\to L^2(\hat{K})$ and $\Pi_{\partial\hat{K}}$: $H^2(\hat{K})\to L^2(\partial\hat{K})$ are continuous, then there exists a constant C independent of h and u_0 such that the $L^2(\Omega)$-error estimate*

$$\|u_0-u_h^0\|_{0,\Omega}\leq C\{h^{r_1+1}\|u_0\|_{r_1+1,\Omega}+h^{r_2+2}\|u_0\|_{r_2+2,\Omega}\} \tag{5.18}$$

is true when u_0 *is in* $H^r(\Omega)\cap H_0^1(\Omega)$.

Remark. First, the mechanical meaning of condition 1) in the theorem is that the states of the zero energy can only be the states of the rigid displacements. Secondly, here we can get the convergence of curved elements only for elliptic equations. To prove the convergence of curved elements for elasticity problems, we must do further work. In this case, the difficulty is that the displacement on element K may not be rigid when the energy is zero on K. We must seek for a new way to reach this destination.

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