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ASYMPTOTIC EXPANSIONS OF THE CUBIC SPLINE COLLOCATION SOLUTION FOR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we consider the following problem

$$\begin{cases} u''(x)=f(x, u(x), u'(x)), & a\leqslant x\leqslant b, \\ u(a)=u(b)=0, \end{cases}$$

and obtain the following theorem.

Theorem. *Suppose that $s(x)$ is a unique collocation solution of the above equation. The solution $u(x)$ of the above equation exists uniquely and $u(x)\in C^{r+2}[a, b]$, $f(x, y, z)\in C^r([a, b]\times(-\infty, \infty)\times(-\infty,\infty))$, $l=\left[\frac{r-1}{2}\right]$, $\frac{\partial f}{\partial y}\geqslant 0$. Then*

$$s(x_i)=u(x_i)+\sum_{j=1}^{l} h^{2j}e_j(x_i)+O(h^r), \quad i=0, 1, \cdots, N,$$

where $e_j(x)$ are solutions of some linear ordinary differential equations.

We consider boundary-value problems of the first kind for second-order differential equations

$$\begin{cases} u''(x)=f(x, u(x), u'(x)), & a\leqslant x\leqslant b, \\ u(a)=u(b)=0, \end{cases} \tag{1}$$

where $u(x)$ is assumed to exist uniquely and $u(x)\in C^{r+2}[a, b]$.

$$f(x, y, z)\in C^r([a, b]\times(-\infty, \infty)\times(-\infty, \infty)), \text{ and } \frac{\partial f(x, y, z)}{\partial y}\geqslant 0.$$

Let Δ be an equidistant partition of $[a, b]$

$$\Delta: \ a=x_0<x_1<\cdots<x_N=b$$

and let $SP(3, \Delta)$ be a cubic spline space.

Definition. *If there is a unique $s(x)\in SP(3, \Delta)$ such that*

$$s(a)=s(b)=0,$$

$$s''(x_i)=f(x_i, s(x_i), s'(x_i)), \quad i=0, 1, \cdots, N,$$

then $s(x)$ is called the cubic spline collocation solution of boundary value problems of (1).

In the following, we always assume that collocation solutions exist uniquely.

Let
$$s(x)=\sum_{i=-1}^{N+1}\alpha_i B_i(x),$$

where $B_i(x)$ are B-Splines of a cubic spline. By using the values of $B_i(x_j)$, $B_i'(x_j)$,

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$B_i''(x_j)$ we have a system of nonlinear equations.

$$\begin{cases} \alpha_{-1}+4\alpha_0+\alpha_1=0, \\ \dfrac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{h^2}=f\left(x_i,\ \dfrac{\alpha_{i+1}+4\alpha_i+\alpha_{i-1}}{6},\ \dfrac{\alpha_{i+1}-\alpha_{i-1}}{2h}\right), \quad i=0,\ 1,\ \cdots,\ N, \\ \alpha_{N-1}+4\alpha_N+\alpha_{N+1}=0. \end{cases}$$

Cancelling α_{-1}, α_{N+1} we obtain

$$\begin{cases} -6\alpha_0=h^2f\left(x_0,\ u_0,\ \dfrac{\alpha_1+2\alpha_0}{h}\right), \\ \dfrac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{h^2}=f\left(x_i,\ \dfrac{\alpha_{i+1}+4\alpha_i+\alpha_{i-1}}{6},\ \dfrac{\alpha_{i+1}-\alpha_{i-1}}{2h}\right), \quad i=1,\ 2,\ \cdots,\ N-1, \\ -6\alpha_N=h^2f\left(x_N,\ u_N,\ -\dfrac{\alpha_{N-1}+2\alpha_N}{h}\right). \end{cases} \tag{2}$$

Solving this system by Newton's iteration we can obtain the collocation solution $s(x)$.

For simplicity let us assume

$$l=\left[\frac{r-1}{2}\right] \quad \text{and} \quad 0!=\frac{1}{3}.$$

For any function $d_j(x)\in C^{r+2-2j}[a,\ b]$ $(j=0,\ 1,\ \cdots,\ l)$, if we take $\beta_i=\sum_{j=0}^{l}h^{2j}d_j(x_i)$ $(i=0,\ 1,\ \cdots,\ N)$, then for $i=1,\ 2,\ \cdots,\ N-1$, from Taylor's formula we have

$$\frac{\beta_{i+1}-2\beta_i+\beta_{i-1}}{h^2}=\sum_{j=0}^{l}h^{2j}\sum_{k=0}^{j}\frac{2\cdot d_{j-k}^{(2k+2)}(x_i)}{(2k+2)!}+O(h^r),$$

$$\frac{\beta_{i+1}+4\beta_i+\beta_{i-1}}{6}=\sum_{j=0}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_i)}{3\cdot(2k)!}+O(h^r),$$

$$\frac{\beta_{i+1}-\beta_{i-1}}{2h}=\sum_{j=0}^{l}h^{2j}\sum_{k=6}^{j}\frac{d_{j-k}^{(2k+1)}(x_i)}{(2k+1)!}+O(h^r),$$

$$\begin{aligned} &f\left(x_i,\ \frac{\beta_{i+1}+4\beta_i+\beta_{i-1}}{6},\ \frac{\beta_{i+1}-\beta_{i-1}}{2h}\right) \\ &=f(x_i,\ d_0(x_i),\ d_0'(x_i))+\sum_{1<s+m<l}\frac{1}{s!m!}\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_i,\ d_0(x_i),\ d_0'(x_i))\Big) \\ &\quad\times\left(\sum_{j=1}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k+1)}(x_i)}{(2k+1)!}+O(h^r)\right)^m \\ &\quad\times\left(\sum_{j=1}^{l}h^{2j}\sum_{k=0}^{j}\frac{d_{j-k}^{(2k)}(x_i)}{3\cdot(2k)!}+O(h^r)\right)^s+O(h^r) \\ &=f(x_i,\ d_0(x_i),\ d_0'(x_i))+\sum_{1<s+m<l}\frac{1}{s!m!}\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_i,\ d_0(x_i),\ d_0'(x_i)) \\ &\quad\times\sum_{j=s+m}h^{2j}\left\{\sum_{\substack{t_1+\cdots+t_{s+m}=j\\ t_r\geq 1}}\prod_{n=1}^{s}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k)}(x_i)}{3\cdot(2k)!}\right)\prod_{n=s+1}^{s+m}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k+1)}(x_i)}{(2k+1)!}\right)\right\}+O(h^r) \\ &=f(x_i,\ d_0(x_i),\ d_0'(x_i))+\sum_{j=1}^{l}h^{2j}\sum_{1<s+m<j}\frac{1}{s!m!}\frac{\partial^{s+m}}{\partial y^s\partial z^m}f(x_i,\ d_0(x_i),\ d_0'(x_i)) \\ &\quad\times\left\{\sum_{\substack{t_1+\cdots+t_{s+m}=j\\ t_r\geq 1}}\prod_{n=1}^{s}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k)}(x_i)}{3\cdot(2k)!}\right)\cdot\prod_{n=s+1}^{s+m}\left(\sum_{k=0}^{t_n}\frac{d_{t_n-k}^{(2k+1)}(x_i)}{(2k+1)!}\right)\right\}+O(h^r), \end{aligned} \tag{3}$$