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A BALANCED OVERSAMPLING FINITE ELEMENT METHOD FOR ELLIPTIC PROBLEMS WITH OBSERVATIONAL BOUNDARY DATA *

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Abstract

In this paper we propose a finite element method for solving elliptic equations with observational Dirichlet boundary data which may subject to random noises. The method is based on the weak formulation of Lagrangian multiplier and requires balanced oversampling of the measurements of the boundary data to control the random noises. We show the convergence of the random finite element error in expectation and, when the noise is sub-Gaussian, in the Orlicz ψ_2-norm which implies the probability that the finite element error estimates are violated decays exponentially. Numerical examples are included.

Mathematics subject classification: 65N30, 65D10, 41A15.
Key words: Observational boundary data, Elliptic equation, Sub-Gaussian random variable.

1. Introduction

In many scientific and engineering applications involving partial differential equations (PDEs), the input data such as sources or boundary conditions are usually given through the measurements which may subject to random noises. For example, in the application of seismic imaging/inversion problems, the measurements are the wave field collected on the surface of the ground which are inevitably polluted by random noises. The collected date will serve as the boundary value for the governing wave equation or Helmholtz equation to find the subsurface structure by some imaging or inversion algorithms (see, e.g., [21], [6]). In this paper, as the first effort to design numerical methods which control both the discretization and random noise error simultaneously for solving PDEs with random input data, we consider the Laplace equation with Dirichlet observational boundary data.

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Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. In this paper we consider the problem to find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_0 \quad \text{on } \Gamma. \tag{1.1}$$

Here $f \in L^2(\Omega)$ is given but the boundary condition $g_0 \in H^2(\Gamma)$ is generally unknown. Here we assume the boundary condition $g_0 \in H^2(\Gamma)$ since our measurements are pointwise on the boundary, this is also what one usually encounters in practice. We assume we know the measurements $g_i = g_0(x_i) + e_i$, $i = 1, 2, \cdots, n$, where $\mathbb{T} = \{x_i : 1 \leq i \leq n\}$ is the set of the measurement locations on the boundary Γ and e_i, $i = 1, 2, \cdots, n$, are independent identically distributed random variables over some probability space $(\mathfrak{X}, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}[e_i] = 0$ and $\mathbb{E}[e_i^2] = \sigma^2 > 0$. In this paper $\mathbb{P}$ denotes the probability measure and $\mathbb{E}[X]$ denotes the expectation of the random variable X. We remark that for simplicity we only consider the problem of observational Dirichlet boundary data in this paper and the problem with observational sources f or other type of boundary conditions can be studied by the same method. We also remark that the extension of the results in this paper to higher dimension PDEs is an interesting problem worth of future investigations.

The classical problem to find a smooth function from the knowledge of its observation at scattered locations subject to random noises is well studied in the literature [24]. One popular model to tackle this classical problem is to use the thin plate spline model [11, 22] which can be efficiently solved by using finite element methods [1, 7, 17]. The scattered data in our problem (1.1) are defined on the boundary of the domain and a straightforward application of the method developed in [1, 7, 11, 17, 22] would lead to solving a fourth order elliptic equation on the boundary which would be much more expansive than the method proposed in this paper.

If the boundary condition g_0 is smooth and it does not have noise, the standard finite element method to solve (1.1) would be to enforce the Dirichlet boundary condition nodewise on the boundary. If the boundary data contain noise, however, the aforementioned standard finite element method does not work and our crucial finding is that one must enforce the Dirichlet condition in some weak sense. Our method is based on the following weak formulation of Lagrangian multiplier for (1.1) in [2]: Find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$(\nabla u, \nabla v) + \langle \lambda, v \rangle = (f, v), \quad \forall v \in H^1(\Omega), \tag{1.2}$$

$$\langle \mu, u \rangle = \langle \mu, g_0 \rangle, \quad \forall \mu \in H^{-1/2}(\Gamma), \tag{1.3}$$

where $(\cdot, \cdot)$ is the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$ which is an extension of the inner product of $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ which is an extension of the inner product of $L^2(\Gamma)$. We remark that while the method of Lagrangian multiplier is one of the standard ways in enforcing Dirichlet boundary condition on smooth domains, it is essential here for solving the problem with Dirichlet observational boundary data even when the domain Ω is a polygon. One may apply the techniques developed in this paper to other weak formulations such as mixed finite element method or those in [18] to deal with observational Dirichlet boundary data.

Let Ω_h be a polygonal domain which approximates the domain Ω. Let $V_h \subset H^1(\Omega_h)$ and $Q_h \subset L^2(\Gamma)$ be the finite element spaces for approximating the field variable and the Lagrangian multiplier. Our finite element method is defined as follows: Find $(u_h, \lambda_h) \in V_h \times Q_h$ such that

$$(\nabla u_h, \nabla v_h)_{\Omega_h} + \langle \lambda_h, v_h \rangle_n = (I_h f, v_h)_{\Omega_h}, \quad \forall v_h \in V_h,$$

$$\langle \mu_h, u_h \rangle_n = \langle \mu_h, g \rangle_n, \quad \forall \mu_h \in Q_h,$$