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PIECEWISE SPARSE RECOVERY VIA PIECEWISE INVERSE SCALE SPACE ALGORITHM WITH DELETION RULE*

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Abstract

In some applications, there are signals with piecewise structure to be recovered. In this paper, we propose a piecewise_ISS (P_ISS) method which aims to preserve the piecewise sparse structure (or the small-scaled entries) of piecewise signals. In order to avoid selecting redundant false small-scaled elements, we also implement the piecewise_ISS algorithm in parallel and distributed manners equipped with a deletion rule. Numerical experiments indicate that compared with aISS, the P_ISS algorithm is more effective and robust for piecewise sparse recovery.

Mathematics subject classification: 90C25, 94A12.
Key words: Inverse scale space, Piecewise sparse, Sparse recovery, Small-scaled entries.

1. Introduction

In this paper, we consider recovering a sparse signal $\mathbf{x}^* \in \mathbf{R}^n$ from its noisy linear measurements

$$\mathbf{b} = A\mathbf{x}^* + \mathbf{e}, \tag{1.1}$$

where $\mathbf{b} \in \mathbf{R}^m$ is a measurement vector, $A \in \mathbf{R}^{m\times n}$ is a measurement matrix, and $\mathbf{e} \in \mathbf{N}(0, \sigma^2\mathbf{I}_n)$ is Gaussian noise. The sparse vector $\mathbf{x}^*$ has $s \le m < n$ nonzero entries. A widely used method to perform this reconstruction is the Basis Pursuit, i.e., to solve the following minimization problem

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \qquad s.t. A\mathbf{x} = \mathbf{b}. \tag{1.2}$$

The key of recovering a signal in this setting is to find the support of the signal, i.e., find the set $\mathbf{S}$ satisfing $supp(\mathbf{x}^*) = \mathbf{S}$, it is named as "exact support recovery". In some applications, the signal is indeed "piecewise sparse". For example, the problem of the decomposition of texture part and cartoon part of image in [20], i.e., $\mathbf{b} = A_n\mathbf{x}_n + A_t\mathbf{x}_t$ where n and t represent the cartoon and texture. It is assumed that both parts can be represented in some given dictionaries, thus $\mathbf{x}_n$ and $\mathbf{x}_t$ are two sparse vectors. The coefficient vector $\mathbf{x} = (\mathbf{x}_n^T, \mathbf{x}_t^T)^T$ is "piecewise" sparse vector. Another example is the problem of reconstructing a surface from scattered data in approximation space $H = \bigcup_{i=1}^N H_j$, where $H_j \subseteq H_{j+1}$ are principal shift invariant (PSI) spaces generated by a single compactly supported function [18], the fitting surface is $g = \sum_{i=1}^N g_i,\ g_i \in H_i$ with $g_i = \sum_{j=1}^{n_i} c_j^i \phi_j^i$. The coefficients $\mathbf{c} = (\mathbf{c}^1, \dots, \mathbf{c}^N)^T$ (by N

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pieces $\mathbf{c}^i = (c_1^i, \ldots, c_{n_i}^i)^T$) is the vector to be determined. Due to the property of PSI spaces, the coefficients to be determined by l^1 minimization in [18] are "piecewise" sparse structured, i.e., each $\mathbf{c}^i \in \mathbf{R}^{n_i}$ is a sparse vector in H_i.

To be general, we recover a sparse signal $\mathbf{x} = (\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)^T$ which is piecewise sparse structured by a partition of support set $\mathbf{S} = (S_i)_{i=1}^N$. Denote the corresponding partition of $\mathbf{D} = \{1, \ldots, n\}$ as $\mathbf{D} = (D_i)_{i=1}^N$. It is clear that $S_i \subseteq D_i$. Then we recover N sub-signals $\mathbf{x}_i$ ($\mathbf{x}_i \in \mathbf{R}^{n_i}$ is s_i-sparse vector on set D_i, where $s_i = |S_i|$) for $i = 1, \ldots, N$, respectively and simultaneously. We call this type of signal as "piecewise sparse" vector, denoted by $(s_1, \ldots, s_N)$-sparse vector. According to the piecewise structure of the signal $\mathbf{x}$, the measurement matrix A is also structured as $A = [A_1, \ldots, A_N]$ where $A_i \in \mathbf{R}^{m \times n_i}$. Then the linear measurements (1.1) can be rewritten as

$$\mathbf{b} = \sum_{i=1}^{N} A_i \mathbf{x}_i^* + \mathbf{e}.$$

Based on this, we provide the definition of piecewise sparse vector:

Definition 1.1. *Suppose the m-sample vector $\mathbf{b}$ is the linear superposition of N components with some additive noise,*

$$\mathbf{b} = \sum_{i=1}^{N} \mathbf{b}_i + \mathbf{e}. \tag{1.3}$$

Furthermore, assume that each $\mathbf{b}_i$ can be sparsely represented in a basis A_i, i.e.,

$$\mathbf{b}_i = A_i \mathbf{x}_i, \ \ i = 1, \ldots, N,$$

where $\mathbf{x}_i$ is a sparse vector. We define the vector $\mathbf{x} = (\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)^T$ as a piecewise sparse vector. In particular, if the piecewise sparsity is provided, i.e., number of nonzero entries of $\mathbf{x}_i$ is s_i for each i, then we denote the piecewise sparse vector $\mathbf{x} = (\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)^T$ as $(s_1, \ldots, s_N)$-piecewise sparse vector.

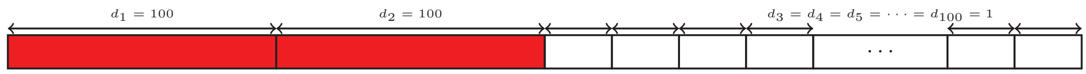

Fig. 1.1. Example of block sparse vector.

Remark 1.1. It is necessary to claim that the piecewise sparse vector is quite different from the block sparse vector mentioned in [14–16,26]. A block s-sparse vector $x = (x^T[1], \ldots, x^T[N])^T$ is assumed to have at most s blocks with nonzero entries while each block $x[l]$ ($l = 1, \ldots, N$) is not necessary sparse. Furthermore, a block sparse vector is not necessary sparse. See the example in [16] (Fig. 1.1). In this example, 2 nonzero blocks out of 100 blocks correspond to 200 nonzero elements out of 298 elements.A piecewise sparse vector $\mathbf{x} = (\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T)^T$ is partitioned into N components and it is assumed that every $\mathbf{x}_i \in \mathbf{R}^{n_i}$ containing nonzero entries is sparse. See the following example in Fig. 1.2, there are 100 parts are each part has one nonzero element. It is clear that a piecewise sparse vector must be a sparse vector in general meaning.

Remark 1.2. Note that the sub-vectors $\mathbf{x}_i^*$ ($i = 1, \ldots, N$) in equation $\mathbf{b}^\sigma = \sum_{i=1}^N A_i \mathbf{x}_i^* + \mathbf{e}$ are correlated to each other, thus these sub-vectors $\mathbf{x}_i^*$ ($i = 1, \ldots, N$) cannot be recovered independently.