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A CELL-CENTERED MULTIGRID SOLVER FOR THE FINITE VOLUME DISCRETIZATION OF ANISOTROPIC ELLIPTIC INTERFACE PROBLEMS ON IRREGULAR DOMAINS*

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Abstract

The aim of this paper is to develop a fast multigrid solver for interpolation-free finite volume (FV) discretization of anisotropic elliptic interface problems on general bounded domains that can be described as a union of blocks. We assume that the curved interface falls exactly on the boundaries of blocks. The transfinite interpolation technique is applied to generate block-wise distorted quadrilateral meshes, which can resolve the interface with fine geometric details. By an extensive study of the harmonic average point method, an interpolation-free nine-point FV scheme is then derived on such multi-block grids for anisotropic elliptic interface problems with non-homogeneous jump conditions. Moreover, for the resulting linear algebraic systems from cell-centered FV discretization, a high-order prolongation operator based fast cascadic multigrid solver is developed and shown to be robust with respect to both the problem size and the jump of the diffusion coefficients. Various non-trivial examples including four interface problems and an elliptic problem in complex domain without interface, all with tens of millions of unknowns, are provided to show that the proposed multigrid solver is dozens of times faster than the classical algebraic multigrid method as implemented in the code AMG1R5 by Stüben.

Mathematics subject classification: 65N08, 65N55.
Key words: Elliptic interface problem, Discontinuous coefficients, Anisotropic coefficients, Cascadic multigrid method, Richardson extrapolation.

1. Introduction

In this paper we consider the following anisotropic elliptic interface problem:

$$-\nabla\cdot(\kappa\nabla u)=f \quad \text{in } \Omega^{+}\cup\Omega^{-}, \tag{1.1}$$

$$u=u_D \quad \text{on } \partial\Omega, \tag{1.2}$$

$$[u]=v \quad \text{on } \Gamma, \tag{1.3}$$

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$$[\kappa \nabla u \cdot \mathbf{n}] = w \quad \text{on } \Gamma \tag{1.4}$$

on a bounded domain $\Omega = \Omega^+ \cup \Gamma \cup \Omega^- \in \mathbb{R}^2$. The subdomains Ω^+ and Ω^- are separated by the interface $\Gamma = \Omega^+ \cap \Omega^-$. The diffusion coefficient κ is a 2×2 symmetric positive definite matrix whose eigenvalues satisfy $\lambda_i \geq \lambda_0 > 0, i = 1, 2$, κ^+ and κ^- are restrictions of κ on Ω^+ and Ω^-, respectively, that is,

$$\kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{12} & \kappa_{22} \end{pmatrix} = \begin{cases} \kappa^+, & \in \Omega^+, \\ \kappa^-, & \in \Omega^-, \end{cases}$$

in which κ may be discontinuous across the interface and causes a low global regularity of u, $f \in C(\Omega \setminus \Gamma)$ is a source term. The two jump conditions

$$v = u^+ - u^-, \quad w = \kappa^+ \nabla u^+ \cdot \mathbf{n} - \kappa^- \nabla u^- \cdot \mathbf{n}$$

are known functions across the interface Γ with at least C^1 continuity, and $\mathbf{n}$ is the unit normal direction pointing to the outer boundary (see Fig. 1.1). When $v = w = 0$ and the diffusion coefficient is continuous ($\kappa^+ = \kappa^-$), the problem degenerates into an anisotropic elliptic boundary value problem.

This type of problems occurs widely in the modeling of many physical phenomena such as crystal growths, Hele-Shaw flows, etc. Mathematically, it usually leads to partial differential equations with discontinuous or non-smooth solutions across interfaces. Hence, classical numerical methods designed for smooth solutions do not work efficiently. And many new methods have been developed in the last four decades, which can be roughly classified into two categories by using either an unfitted mesh (e.g. a uniform Cartesian mesh) or an interface-fitted (also known as body-fitted or interface conforming) mesh in the discretization of the domain.

In the unfitted mesh approach, the numerical approximation methods are modified locally near the interface to satisfy the jump conditions, of which two typical ones are the immersed interface method [31] and immersed finite element method [32]. The most attractive feature of the unfitted mesh approach is the easiness of the mesh generation, which is convenient for moving interface problems. Recent progress on the unfitted mesh approach can be found in [16, 18].

In this work, we focus on the interface-fitted mesh approach, which can be divided into three types: completely unstructured grid, locally unstructured grid, and semi-structured grid. The completely unstructured mesh generator is time consuming as it needs to modify the mesh for the whole domain, not just near the interface. The locally unstructured mesh generator [9] modifies just the mesh near the interface thus is more efficient than that of a completely

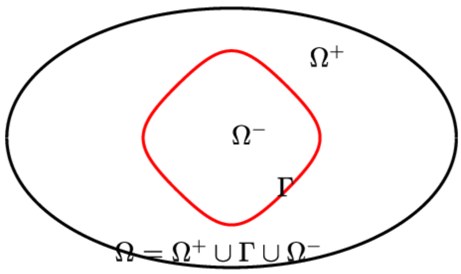

Fig. 1.1. An illustration of a general bounded domain with a curved interface.