Journal of Computational Mathematics
Vol.43, No.1, 2025, 43–62.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2307-m2022-0225

# STABLE RECOVERY OF SPARSE SIGNALS WITH NON-CONVEX WEIGHTED $r$-NORM MINUS 1-NORM*

Jianwen Huang
*School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China;*
*School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China*
*Email: hjw1303987297@126.com*
Feng Zhang[1)]
*School of Mathematics and Statistics, Southwest University, Chongqing 400715, China*
*Email: zfmath@swu.edu.cn*
Xinling Liu
*Key Laboratory of Optimization Theory and Applications at China West Normal University of Sichuan Province, School of Mathematics and Information, China West Normal University, Nanchong 637009, China*
*Email: fsliuxl@163.com*
Jianjun Wang
*School of Mathematics and Statistics, Southwest University, Chongqing 400715, China;*
*School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China*
*Email: wjj@swu.edu.cn*
Jinping Jia
*School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China*
*Email: jjping2006@163.com*
Runke Wang
*College of Resources and Environmental Engineering, Tianshui Normal University, Tianshui 741001, China*
*Email: wrk0531@tsnu.edu.cn*

**Abstract**

Given the measurement matrix $A$ and the observation signal $y$, the central purpose of compressed sensing is to find the most sparse solution of the underdetermined linear system $y = Ax + z$, where $x$ is the $s$-sparse signal to be recovered and $z$ is the noise vector. Zhou and Yu [Front. Appl. Math. Stat., 5 (2019), Article 14] recently proposed a novel non-convex weighted $\ell_r - \ell_1$ minimization method for effective sparse recovery. In this paper, under newly coherence-based conditions, we study the non-convex weighted $\ell_r - \ell_1$ minimization in reconstructing sparse signals that are contaminated by different noises. Concretely, the results reveal that if the coherence $\mu$ of measurement matrix $A$ fulfills

$$\mu < \kappa(s; r, \alpha, N), \quad s > 1, \quad \alpha^{\frac{1}{r}} N^{\frac{1}{2}} < 1,$$

then any $s$-sparse signals in the noisy scenarios could be ensured to be reconstructed robustly by solving weighted $\ell_r - \ell_1$ minimization non-convex optimization problem. Furthermore, some central remarks are presented to clear that the reconstruction assurance is much weaker than the existing ones. To the best of our knowledge, this is the first mutual coherence-based sufficient condition for such approach.

*Mathematics subject classification:* 68W40, 68P30, 94A08, 94A12.
*Key words:* Compressed sensing, Sparse recovery, Mutual coherence, Sufficient condition.

[*] Received October 10, 2022 / Revised version received February 12, 2023 / Accepted July 4, 2023 / Published online December 14, 2023 /

[1)] Corresponding author

# 1. Introduction

Compressed sensing [3, 8] has recently triggered much interest in signal and imaging processing, statistics and applied mathematics. The crucial aim is to recover a high-dimensional sparse signal from a small quantity of linear measurements. Generally, one thinks about the linear model

$$y = Ax + z, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times N}$ is the measurement matrix with $m \ll N$, $z \in \mathbb{R}^m$ is the noise vector and $x \in \mathbb{R}^N$ is an $s$-sparse (i.e. the number of nonzero elements of $x$ is not more than $s$) vector to be recovered. Two widely utilized types of noises are the bounded $\ell_2$ noise [9, 15] and the Dantzig selector noise [4], respectively. Throughout the article, we suppose that the columns of $A$ are standardized, i.e. for all $i$, $A_i^\top A_i = 1$, where $A_i, 1 \le i \le N$, denotes the $i$-th column of $A$.

Because the linear model (1.1) is an underdetermined linear system, it is impossible to stably reconstruct $x$ based on $A, z$ and $y$. Fortunately, it is possible to stably reconstruct $s$-sparse signal $x$ from (1.1) with a few appropriately exploiting sparse reconstruction methods under suitable assumptions regarding $A$ and $z$. There are two extensively applied frameworks to describe such assumptions concerning $A$, which are separately the restricted isometry property (RIP) [3] and the mutual coherence determined as [10, 12]

$$\mu = \max_{1\le i<j\le N} |\langle A_i, A_j\rangle|. \tag{1.2}$$

For a more general definition of coherence, i.e. the block-coherence, see [13].

It is well known that $\ell_1$ minimization method [7], viewed as a convex extension of $\ell_0$ minimization method, presents an efficient approach for recovering $s$-sparse signal in numerous contexts. The $\ell_0$ minimization method and the $\ell_1$ minimization method are respectively

$$\begin{aligned} &\min_{\tilde{x}\in\mathbb{R}^N} \|\tilde{x}\|_0 \\ &\text{s.t.} \quad y = A\tilde{x} + z, \quad \|z\|_2 \le \epsilon, \\ &\min_{\tilde{x}\in\mathbb{R}^N} \|\tilde{x}\|_1 \\ &\text{s.t.} \quad y = A\tilde{x} + z, \quad \|z\|_2 \le \epsilon. \end{aligned}$$

Here $\|\tilde{x}\|_0$ represents the number of non-zero coordinates in $\tilde{x}$. In recent years, one alternative approach of estimating the $s$-sparse signal in the references [6, 11, 14, 24, 25, 29] is to solve the following $\ell_r$ minimization model:

$$\begin{aligned} &\min_{\tilde{x}\in\mathbb{R}^N} \|\tilde{x}\|_r^r \\ &\text{s.t.} \quad y = A\tilde{x} + z, \quad \|z\|_2 \le \epsilon, \end{aligned} \tag{1.3}$$

where $\|\tilde{x}\|_r^r = \sum_{i=1}^N |\tilde{x}_i|^r$ with $\tilde{x}_i$ being the $i$-th entry of $\tilde{x}, r \in (0, 1]$ and

$$\|z\|_2 = \left(\sum_{i=1}^m z_i^2\right)^{\frac{1}{2}}.$$

Compared with $\ell_1$ minimization, although it is more difficult to resolve model (1.3) because of its noncovexity, there still exist a lot of algorithms to find the local optimal solution of (1.3).