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OPTIMAL ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD WITH GENERALIZED NUMERICAL FLUXES FOR ONE-DIMENSIONAL KDV TYPE EQUATIONS*

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Abstract

In this paper, we investigate the local discontinuous Galerkin method with generalized numerical fluxes for one-dimensional nonlinear Korteweg-de Vries type equations. The numerical flux for the nonlinear convection term is chosen as the generalized Lax-Friedrichs flux, and the generalized alternating flux and upwind-biased flux are used for the dispersion term. The generalized Lax-Friedrichs flux with anti-dissipation property will compensate the numerical dissipation of the dispersion term, resulting in a nearly energy conservative scheme that is useful in resolving waves and is beneficial for long time simulations. To deal with the nonlinearity and different numerical flux weights, a suitable numerical initial condition is constructed, for which a modified global projection is designed. By establishing relationships between the prime variable and auxiliary variables in combination with sharp bounds for jump terms, optimal error estimates are obtained. Numerical experiments are shown to confirm the validity of theoretical results.

Mathematics subject classification: 65M12, 65M15, 65M60.
Key words: Korteweg-de Vries type equations, Local discontinuous Galerkin method, Generalized fluxes, Error estimates.

1. Introduction

In this paper, we study the local discontinuous Galerkin (LDG) method with generalized numerical fluxes for one-dimensional nonlinear Korteweg-de Vries (KdV) type equations

$$u_t + f(u)_x + u_{xxx} = 0, \quad (x,t) \in I \times (0,T], \tag{1.1a}$$
$$u(x,0) = u_0(x), \qquad x \in I, \tag{1.1b}$$

where $u_0(x)$ is a smooth function and $I = [a,b]$. The nonlinear function $f(u)$ is assumed to be sufficiently smooth with respect to u, and the exact solution u is smooth. The periodic boundary conditions are mainly considered, and the case with mixed boundary conditions is numerically investigated. For KdV equations, compared with the standard upwind and alternating fluxes, the energy conserving scheme will produce a lower growth of errors and is efficient in resolving waves. This can be achieved by choosing central fluxes for generalized KdV equations [1] or the generalized numerical fluxes with different weights for linearized KdV equations [11]. For

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nonlinear KdV type equations (1.1), by constructing a suitable numerical initial condition and a modified projection in combination with the relationships between the prime variable and auxiliary variables, optimal error estimates are derived.

The nonlinear KdV type equation is an important model for many nonlinear phenomena, which can describe wave phenomena in bubble-liquid mixtures [19], plasma physics [8] and anharmonic crystals [24]. There have been a variety of work on the theoretical and numerical aspects of KdV equations. For example, in [7], a generalized tanh function method was implemented to find the exact solutions of the KdV equation and the coupled KdV equation. A meshless method of lines was presented for the numerical solution of the KdV equation in [16]. Numerical solution of the KdV equation was obtained using the space-splitting technique and the differential quadrature method with cosine expansion [15].

The LDG method is an extension of the discontinuous Galerkin (DG) method. The DG method is a class of finite element method using discontinuous piecewise polynomials as the numerical solution and test functions, leading to advantages in high order accuracy, high parallel efficiency, flexibility for *hp*-adaptivity. It was first introduced to solve a linear steady-state hyperbolic equation [14] and was developed for solving nonlinear time dependent conservation laws [3,5]. The LDG method was proposed by Cockburn and Shu [4] to solve convection-diffusion equations. The main idea of the LDG method is to rewrite the original partial differential equation (PDE) involving high order spatial derivatives into an equivalent first order system and then the DG method can be applied. Later, it was actively applied to solve various high order equations. We refer to review papers [17, 21] for more details.

The LDG scheme for KdV type equations was first proposed in [23], in which stability property was shown for nonlinear case and suboptimal $(k + 1/2)$-th order was derived for the linear case. In [10], the method was extended to solve the nonlinear dispersive PDE involving compactly supported traveling wave solutions. For the LDG scheme solving nonlinear KdV equations, suboptimal $(k + 1/2)$-th order error estimate was obtained [20], and the loss of half an order is mainly due to some extra boundary terms arising from high order derivatives. By establishing several energy equations, optimal error estimate of order $k + 1$ is derived for linearized KdV equations [22]. Note that purely upwind and alternating fluxes are used in above work. For generalized KdV equations, a posteriori error estimates of conservative LDG methods is given [9]. In [25], for KdV type systems, four conservative and dissipative LDG schemes are proposed, in which the conservative/dissipative numerical fluxes are designed for the linear dispersion term and the nonlinear convection term, respectively. By virtue of some local Gauss-Radau projections, suboptimal error estimates of order $k+1/2$ are derived for dissipative fluxes, and numerical examples indicate that the conservative scheme performs better than the dissipative one for long time simulations.

In addition to the stability issue of the numerical fluxes in the design of scheme, the numerical viscosity plays an important role in resolving waves and for long time simulations. The LDG method with central and generalized alternating fluxes for solving the Burgers-Poisson equation was presented in [13]. The LDG scheme with upwind-biased and generalized alternating fluxes for linear convection-diffusion problem was discussed in [2]. In [11], the LDG method using generalized numerical fluxes for linearized KdV equations was studied and the optimal error estimate was obtained. For scalar nonlinear hyperbolic conservation laws, the generalized local Lax-Friedrichs (GLLF) flux that may not be monotone was proposed and optimal error estimate was shown in [12]. In these studies, according to different choices of numerical fluxes, generalized Gauss-Radau (GGR) projections were proposed, and an analysis