Journal of Computational Mathematics
Vol.43, No.1, 2025, 89–120.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2308-m2023-0075

SPACE-TIME CONTINUOUS AND TIME DISCONTINUOUS GALERKIN SCHEMES BASED ON ISOGEOMETRIC ANALYSIS FOR NONLINEAR TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

Ang Ge and Jinye Shen[1)]
School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China
Emails: geangswfue@163.com, jyshen@swufe.edu.cn
Lijun Yi
Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
Email: ylj5152@shnu.edu.cn

Abstract

This paper presents space-time continuous and time discontinuous Galerkin schemes for solving nonlinear time-fractional partial differential equations based on B-splines in time and non-uniform rational B-splines (NURBS) in space within the framework of Isogeometric Analysis. The first approach uses the space-time continuous Petrov-Galerkin technique for a class of nonlinear time-fractional Sobolev-type equations and the optimal error estimates are obtained through a concise equivalence analysis. The second approach employs a generalizable time discontinuous Galerkin scheme for the time-fractional Allen-Cahn equation. It first transforms the equation into a time integral equation and then uses the discontinuous Galerkin method in time and the NURBS discretization in space. The optimal error estimates are provided for the approach. The convergence analysis under time graded meshes is also carried out, taking into account the initial singularity of the solution for two models. Finally, numerical examples are presented to demonstrate the effectiveness of the proposed methods.

Mathematics subject classification: 65M12, 65M22, 65M60.
Key words: Space-time, Nonlinear time-fractional Sobolev-type equations, Time-fractional Allen-Cahn equation, Isogeometric analysis, Error estimation.

1. Introduction

In this paper, we consider a class of nonlinear time-fractional Sobolev-type equations (TFSEs)

$$\begin{cases} u_t = {}_0^C D_t^{\alpha} \Delta_{\mathbf{x}} u + g(u), & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u|_{\partial\Omega} = 0, & t \in [0, T], \end{cases} \tag{1.1}$$

and time-fractional Allen-Cahn equation (TFACE)

$$\begin{cases} {}_0^C D_t^{\alpha} u = \epsilon^2 \Delta_{\mathbf{x}} u - f(u), & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u|_{\partial\Omega} = 0, & t \in [0, T], \end{cases} \tag{1.2}$$

* Received April 6, 2023 / Revised version received June 19, 2023 / Accepted August 9, 2023 / Published online December 19, 2023 /
1) Corresponding author

where the Caputo fractional derivative operator ${}_a^C D_t^\alpha$ with $0 < \alpha < 1$ on interval $[a, b] \subset \mathbb{R}$ is defined as follows:

$$ {}_a^C D_t^\alpha u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \frac{\partial u(\mathbf{x}, s)}{\partial s} ds, \quad t \in [a, b]. $$

Here $f(u) = F^{'}(u)$, where $F(u) = (u^2 - 1)^2/4$ is double-well potential function. In addition, Ω is a bounded domain in $\mathbb{R}^d$ with $d = 1, 2, 3$, and

$$ \Delta_{\mathbf{x}} u(\mathbf{x}, t) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}. $$

Caputo derivative can be regarded as $1 - \alpha$ fractional integral of $u_t(\mathbf{x}, t)$, i.e.

$$ {}_a^C D_t^\alpha u(\mathbf{x}, t) = {}_a I_t^{1-\alpha} u_t(\mathbf{x}, t). $$

The Riemann-Liouville fractional integral operator ${}_a I_t^{1-\alpha}$ is defined by

$$ {}_a I_t^{1-\alpha} u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} u(\mathbf{x}, s) ds, \quad t \in [a, b]. \tag{1.3} $$

We assume that nonlinear term $g(u)$ satisfies the following Lipschitz condition: There is a constant $L_1 > 0$ such that

$$ |g(u) - g(v)| \leq L_1 |u - v|. $$

Sobolev equation is a widely used model in fluid mechanics and heat conduction problems [4, 9, 51]. In recent years, there has been ongoing development of effective numerical algorithms for nonlinear Sobolev problem, as seen in works such as [8, 14, 47]. For the TFSEs (1.1), previous studies such as [33, 36, 59] employed the difference method in time direction in combination with spatial finite element or finite volume element discretization to consider the linear form. On the other hand, TFACE model is an important phase field model, with its classical case firstly introduced by Allen and Cahn [2]. Several recent works have focused on developing effective numerical methods to solve problem (1.2), utilizing difference techniques in time direction and introducing stable numerical schemes [19, 23, 32, 49]. Finite difference methods indeed offer several advantages, including simplicity of implementation, computational efficiency, and broad applicability. However, it is important to acknowledge that their accuracy in the temporal direction for fractional problems may be limited, especially when dealing with singularity problems. Therefore, in order to improve accuracy and better handle singular solutions, instead of traditional finite element methods (FEM) combined with difference methods, also called method of lines, we propose to construct time-stepping space-time methods for nonlinear problems (1.1) and (1.2), where temporal direction is discretized by using continuous Petrov-Galerkin (CPG) [21, 22] and discontinuous Galerkin (DG) [29] methods, with solving carried out one by one time slice. Specially, when u is absolutely continuous with respect to t on $[0, T]$, we utilize the semigroup property of the fractional operator ${}_0 I_t^\alpha$ to transform the Eq. (1.2) into an equivalent form [10, Lemma 2.2]

$$ \begin{cases} u = u_0 + \epsilon^2 {}_0 I_t^\alpha \Delta_{\mathbf{x}} u - {}_0 I_t^\alpha f(u), & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u|_{\partial\Omega} = 0, & t \in [0, T]. \end{cases} \tag{1.4} $$

So we construct numerical method for equivalent form (1.4) to obtain the numerical approximation of the solution to original equation (1.2). In some sense, we unify two models as Volterra integro-differential equations with respect to different variables and nonlinear terms.