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TWO FAMILIES OF n-RECTANGLE NONCONFORMING FINITE ELEMENTS FOR SIXTH-ORDER ELLIPTIC EQUATIONS*

Xianlin Jin and Shuonan Wu[1)]
School of Mathematical Sciences, Peking University, Beijing 100871, China
Emails: xianlincn@pku.edu.cn, snwu@math.pku.edu.cn

Abstract

In this paper, we propose two families of nonconforming finite elements on n-rectangle meshes of any dimension to solve the sixth-order elliptic equations. The unisolvent property and the approximation ability of the new finite element spaces are established. A new mechanism, called the exchange of sub-rectangles, for investigating the weak continuities of the proposed elements is discovered. With the help of some conforming relatives for the H^3 problems, we establish the quasi-optimal error estimate for the triharmonic equation in the broken H^3 norm of any dimension. The theoretical results are validated further by the numerical tests in both 2D and 3D situations.

Mathematics subject classification: 65N30.
Key words: Nonconforming finite element method, n-Rectangle element, Sixth-order elliptic equation, Exchange of sub-rectangles.

1. Introduction

Sixth-order partial differential equations have been widely used to model various physical laws and dynamics in material sciences and phase field problems [6, 11]. Owning such a significance in these areas, however, methods for solving the sixth-order equations are insufficient and less studied compared with the lower-order equations from both theoretical and numerical aspects. From a practical point of view, nonconforming finite element method is one of the frequently desired numerical methods for high order partial differential equations. In terms of solving sixth-order equations, the usage of nonconforming spaces allows us to avoid the requirement of C^2-continuity which causes high complexity for the implementation. Having a smaller set of degrees of freedom (DoFs) and a shrunken space of shape functions, yet the nonconforming finite elements should conceivably possess some basic weak continuity properties [20] to preserve the convergence of the numerical solutions. Therefore, the design of such exquisite finite element spaces can be challenging for certain problems, especially in high dimensional situations.

Starting from the solving of fourth-order equations, there are several well-known nonconforming finite elements like the Morley element and the Zienkiewicz element designed on two-dimensional simplicial meshes. A similar idea was then applied to high dimensional case [21], which generalizes the Zienkiewicz element to n-dimensional simplexes where $n \geq 2$. Further, Wang and Xu [23] proposed a family of nonconforming finite elements on simplexes named by the Morley-Wang-Xu element to solve $2m$-th-order elliptic equations where $m \leq n$. This result has been extended to $m = n + 1$ in [25], and to arbitrary m, n with interior stabilization [24].

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1) Corresponding author

Restricted to the low-dimensional cases, the nonconforming finite element spaces for H^2 can be seen in [7], and for H^3 or higher regularity can be found in [17, 18].

On the simplicial meshes, other types of discretization besides the nonconforming finite element method for sixth-order partial differential equations may also be feasible. In two-dimensional case, the H^3 conforming finite element was constructed in [26] and can be generalized to arbitrary H^m [3]. Recently, a construction of conforming finite element spaces with arbitrary smoothness in any dimension was given in [14]. Others include mixed methods [10,19], C^0 interior penalty discontinuous Galerkin method [12], recovery-based method [13], and virtual element methods [8].

As for rectangle meshes, successful constructions of finite element such as the Adini element [1] of C^0 smoothness and Bogner-Fox-Schmidt element (BFS, [2]) of C^1 smoothness were made on two-dimensional grids, whose DoFs are all defined on vertices of rectangles. After an extension [22] to the n-rectangle meshes of any high dimensional spaces where $n \geq 2$, the Adini element and the BFS element possess only C^0 smoothness, and yet their solvabilities to the fourth-order equations have both been remained. Furthermore, an extended version of the Morley element to the n-rectangle meshes was also reported in [22]. For the biharmonic equation, a new family of n-rectangle nonconforming finite element by enriching the second-order serendipity element was constructed in [27]. For arbitrary smoothness, a family of minimal n-rectangle macro-elements was established in [16].

Wang *et al.* [22] showed that the Morley, Adini and BFS elements are of the first-order convergence in the energy norm for solving the biharmonic equation. A more delicate analysis proposed in [15] reveals that the Adini element is capable of reaching a second-order convergence in the energy norm and has an optimal second-order convergence in the L^2-norm. It cannot be overlooked that theories of nonconforming finite element methods are well-prepared for the fourth-order equations on a variety of n-rectangle discretizations, yet very little is extended to the solving of sixth-order problems.

In this paper, we develop two families of n-rectangle nonconforming finite elements for sixth-order partial differential equations. Both the two families of elements are constructed by enriching the DoFs of the n-rectangle Adini element [22] and the corresponding shape function space. Following the well-developed projection-averaging strategy [22], we give the definition of the interpolation operator in high dimensional cases for both two families of elements. It can be shown that the shape function spaces are capable of approximating $H^{3+s}(\Omega)$ for any $s \in [0, 1]$ in an arbitrarily high dimension, which are essential to the error estimate afterwards.

Furthermore, analysis of the weak continuity properties usually plays an important role in the investigation of a nonconforming finite element. Reasonably, difficulties brought by the sixth-order differential operator $(-\Delta)^3$ mainly occur when considering the weak continuities of the following second-order derivatives of the finite element function: the tangential-tangential ($\partial_{\tau\tau}$), normal-normal ($\partial_{\nu\nu}$) and tangential-normal ($\partial_{\tau\nu}$) derivatives across the $(n-1)$-dimensional faces of an element T. It is possible to make use of the interpolations of other well-known n-rectangle finite elements to locally estimate the terms of $\partial_{\tau\tau}$ and $\partial_{\nu\nu}$. However, the analysis of $\partial_{\tau\nu}$ is much more complicated than those terms above for both the two families of elements, so that we only consider estimating this term in a more global manner. We therefore propose a new technique called exchange of sub-rectangles to deal with this complicated term. Combining the results of weak continuities and the help of conforming relatives, we complete estimating the consistency error, which gives the final error estimate by applying the well-known Strang's lemma.