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ADAPTIVE VIRTUAL ELEMENT METHOD FOR CONVECTION DOMINATED DIFFUSION EQUATIONS*

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Abstract

In this paper, a robust residual-based a posteriori estimate is discussed for the Streamline Upwind/Petrov Galerkin (SUPG) virtual element method (VEM) discretization of convection dominated diffusion equation. A global upper bound and a local lower bound for the a posteriori error estimates are derived in the natural SUPG norm, where the global upper estimate relies on some hypotheses about the interpolation errors and SUPG virtual element discretization errors. Based on the Dörfler's marking strategy, adaptive VEM algorithm drived by the error estimators is used to solve the problem on general polygonal meshes. Numerical experiments show the robustness of the a posteriori error estimates.

Mathematics subject classification: 65N15, 65N30, 65N50.
Key words: A posteriori estimate, SUPG virtual element method, Convection dominated diffusion equation, Adaptive VEM algorithm.

1. Introduction

A posteriori error analysis of the SUPG virtual element method in the context of stationary convection dominated diffusion equations is studied in this paper. This kind of equations have many important applications, including river and air pollution, fluid flow and fluid heat conduction. Since the weak solution of such problems exhibits different types of layers, the standard numerical method often leads to oscillations in the solution, if these layers are not efficiently resolved by the mesh. Various stabilized schemes for convection dominated diffusion equations have been developed, for examples, SUPG methods [15,24,31], discontinuous Galerkin methods [20, 22], edge stabilization methods [25, 27], and continuous interior penalty (CIP) methods [16, 23].

Since the numerical solution obtained by the SUPG method is often accompanied by spurious oscillations in a vicinity of layers, a posteriori error estimate for convection dominated diffusion equation is necessary and meaningful. There were already a lot of works about this issue. Verfürth [33] derives a posteriori error estimates for convection-diffusion equations with dominant convection and the ratio of the upper and lower bounds depends on the local mesh-Péclet number. In [34, 35], the robust a posteriori error estimates for stationary and nonstationary convection-diffusion equations are studied. All estimators yield global upper and lower bounds for the error measured in a norm that incorporates the standard energy norm and a dual norm of the convective derivative. Based on some hypotheses that relate interpolation errors in different norms and the error of the SUPG approximation a robust residual-based a posteriori estimates in standard SUPG norm is proposed for the SUPG finite element method discretization of stationary convection diffusion reaction equations in [28].

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Recently, the virtual element method has become an attractive research topic as a method to extend classical finite element method to general polygonal meshes. It has been used in a variety of fields, such as discrete fracture network simulation, incompressible miscible displacements in porous media, resistive magnetohydrodynamics and polycrystal composite materials. Since the original introduction of [3], various problems have been solved by the virtual element method so far, for example [1,2,5,10,18,26,37]. The VEM can handle very general polygonal elements with geometrical hanging nodes, because we just treat the hanging nodes as new nodes. Therefore, it is well suited to mesh refinement and adaptive problems, which can help us save a lot of computational cost. For the development and application of the a posteriori error estimate of VEM, a short representative list being [6,8,9,12,17,21,30,36].

There are a few works on SUPG-VEM of convection dominated diffusion equations. Cangiani *et al.* [29] first studied a non-consistent SUPG-VEM of convection dominated diffusion problem. Subsequently, SUPG-stabilized conforming and non-conforming VEMs are presented in [11,13]. The robustness of a priori error estimates for these methods is proved for high Péclet numbers. This shows the efficiency of the SUPG stabilisation. Recently, Beirão da Veiga *et al.* [7] discussed a robustness analysis of the SUPG-stabilized virtual elements for convection diffusion problems. By slightly modifying the SUPG format of [13], they propose a new way to discretize the convection term, which ultimately demonstrates the robustness of the parameters involved in the convergence estimates without requiring sufficiently small mesh sizes.

The a posteriori error estimate of the virtual element method for convection dominated diffusion equations was not reported up to now. Motivated by [28], in this paper we aims to derive a robust residual-based a posteriori error bounds for the SUPG virtual element approximation of convection dominated diffusion equation. Firstly the virtual element space with corresponding degrees of freedom and SUPG-VEM formulation are introduced. Based the hypotheses between the interpolation errors and SUPG virtual element discretization errors, a global upper bound is deduced. Further, a local lower bound for the a posteriori error estimate is derived. Finally, adaptive VEM algorithm drived by the error estimators is introduced and some numerical examples are carried out to verify our theoretical analysis.

The paper is organized as follows. In the next section, the model problem and the SUPG-VEM formulation are introduced. In Section 3, a global upper bound and a local lower bound for the a posteriori error estimate are derived in the convection dominated case. In the last section we perform some numerical experiments to verify the theoretical results by using the adaptive VEM algorithm.

Throughout the paper, for an open bounded domain E, we use the standard notation $|\cdot|_{s,E}$ and $\|\cdot\|_{s,E}$ to denote seminorm and norm, respectively, in the Sobolev space $H^s(E)$, while $(\cdot,\cdot)_{0,E}$ denotes the $L^2(E)$ inner product. When E is the whole domain Ω, the subscript can be omitted. For every integer $n \geq 0, \mathbb{P}_n(E)$ denotes the space of polynomials of degree at most on E. In particular, $\mathbb{P}_{-1}(E) = \{0\}$. C is a generic constant with different value at different places.

2. The Model Problem and VEM Discretization

In this section, we first introduce the model problem and polynomial projections. Then we introduce the virtual element space with the corresponding degrees of freedom and the SUPG-VEM formulation of problem is given. Finally, we give the relevant knowledge of the SUPG stabilization parameter τ_E.