Journal of Computational Mathematics
Vol.43, No.1, 2025, 174–202.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2309-m2021-0366

ADAPTIVE VIRTUAL ELEMENT METHOD FOR CONVECTION DOMINATED DIFFUSION EQUATIONS*

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Abstract

In this paper, a robust residual-based a posteriori estimate is discussed for the Streamline Upwind/Petrov Galerkin (SUPG) virtual element method (VEM) discretization of convection dominated diffusion equation. A global upper bound and a local lower bound for the a posteriori error estimates are derived in the natural SUPG norm, where the global upper estimate relies on some hypotheses about the interpolation errors and SUPG virtual element discretization errors. Based on the Dörfler's marking strategy, adaptive VEM algorithm drived by the error estimators is used to solve the problem on general polygonal meshes. Numerical experiments show the robustness of the a posteriori error estimates.

Mathematics subject classification: 65N15, 65N30, 65N50.
Key words: A posteriori estimate, SUPG virtual element method, Convection dominated diffusion equation, Adaptive VEM algorithm.

1. Introduction

A posteriori error analysis of the SUPG virtual element method in the context of stationary convection dominated diffusion equations is studied in this paper. This kind of equations have many important applications, including river and air pollution, fluid flow and fluid heat conduction. Since the weak solution of such problems exhibits different types of layers, the standard numerical method often leads to oscillations in the solution, if these layers are not efficiently resolved by the mesh. Various stabilized schemes for convection dominated diffusion equations have been developed, for examples, SUPG methods [15,24,31], discontinuous Galerkin methods [20, 22], edge stabilization methods [25, 27], and continuous interior penalty (CIP) methods [16, 23].

Since the numerical solution obtained by the SUPG method is often accompanied by spurious oscillations in a vicinity of layers, a posteriori error estimate for convection dominated diffusion equation is necessary and meaningful. There were already a lot of works about this issue. Verfürth [33] derives a posteriori error estimates for convection-diffusion equations with dominant convection and the ratio of the upper and lower bounds depends on the local mesh-Péclet number. In [34, 35], the robust a posteriori error estimates for stationary and nonstationary convection-diffusion equations are studied. All estimators yield global upper and lower bounds for the error measured in a norm that incorporates the standard energy norm and a dual norm of the convective derivative. Based on some hypotheses that relate interpolation errors in different norms and the error of the SUPG approximation a robust residual-based a posteriori estimates in standard SUPG norm is proposed for the SUPG finite element method discretization of stationary convection diffusion reaction equations in [28].

* Received December 27, 2021 / Revised version received June 19, 2023 / Accepted September 27, 2023 / Published online December 28, 2023 /
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Recently, the virtual element method has become an attractive research topic as a method to extend classical finite element method to general polygonal meshes. It has been used in a variety of fields, such as discrete fracture network simulation, incompressible miscible displacements in porous media, resistive magnetohydrodynamics and polycrystal composite materials. Since the original introduction of [3], various problems have been solved by the virtual element method so far, for example [1, 2, 5, 10, 18, 26, 37]. The VEM can handle very general polygonal elements with geometrical hanging nodes, because we just treat the hanging nodes as new nodes. Therefore, it is well suited to mesh refinement and adaptive problems, which can help us save a lot of computational cost. For the development and application of the a posteriori error estimate of VEM, a short representative list being [6, 8, 9, 12, 17, 21, 30, 36].

There are a few works on SUPG-VEM of convection dominated diffusion equations. Cangiani *et al.* [29] first studied a non-consistent SUPG-VEM of convection dominated diffusion problem. Subsequently, SUPG-stabilized conforming and non-conforming VEMs are presented in [11, 13]. The robustness of a priori error estimates for these methods is proved for high Péclet numbers. This shows the efficiency of the SUPG stabilisation. Recently, Beirão da Veiga *et al.* [7] discussed a robustness analysis of the SUPG-stabilized virtual elements for convection diffusion problems. By slightly modifying the SUPG format of [13], they propose a new way to discretize the convection term, which ultimately demonstrates the robustness of the parameters involved in the convergence estimates without requiring sufficiently small mesh sizes.

The a posteriori error estimate of the virtual element method for convection dominated diffusion equations was not reported up to now. Motivated by [28], in this paper we aims to derive a robust residual-based a posteriori error bounds for the SUPG virtual element approximation of convection dominated diffusion equation. Firstly the virtual element space with corresponding degrees of freedom and SUPG-VEM formulation are introduced. Based the hypotheses between the interpolation errors and SUPG virtual element discretization errors, a global upper bound is deduced. Further, a local lower bound for the a posteriori error estimate is derived. Finally, adaptive VEM algorithm drived by the error estimators is introduced and some numerical examples are carried out to verify our theoretical analysis.

The paper is organized as follows. In the next section, the model problem and the SUPG-VEM formulation are introduced. In Section 3, a global upper bound and a local lower bound for the a posteriori error estimate are derived in the convection dominated case. In the last section we perform some numerical experiments to verify the theoretical results by using the adaptive VEM algorithm.

Throughout the paper, for an open bounded domain E, we use the standard notation $|\cdot|_{s,E}$ and $\|\cdot\|_{s,E}$ to denote seminorm and norm, respectively, in the Sobolev space $H^s(E)$, while $(\cdot,\cdot)_{0,E}$ denotes the $L^2(E)$ inner product. When E is the whole domain Ω, the subscript can be omitted. For every integer $n \geq 0, \mathbb{P}_n(E)$ denotes the space of polynomials of degree at most on E. In particular, $\mathbb{P}_{-1}(E) = \{0\}$. C is a generic constant with different value at different places.

2. The Model Problem and VEM Discretization

In this section, we first introduce the model problem and polynomial projections. Then we introduce the virtual element space with the corresponding degrees of freedom and the SUPG-VEM formulation of problem is given. Finally, we give the relevant knowledge of the SUPG stabilization parameter τ_E.

2.1. The model problem

In this section we consider the following problem:

$$\begin{cases} -\nabla\cdot(\varepsilon\nabla y)+\boldsymbol{\beta}(\boldsymbol{x})\cdot\nabla y+\delta y=f & \text{in } \Omega, \\ y=0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $0<\varepsilon\ll 1$ represents constant diffusion coefficient, $f\in L^2(\Omega)$ is the volume source term and $\delta>0$ is a constant. We assume that $\boldsymbol{\beta}\in[W^1_\infty(\Omega)]^2$ with $\nabla\cdot\boldsymbol{\beta}=0$ is the transport advective field and $\Omega\subset\mathbb{R}^2$ is a polygonal domain with $\Gamma=\partial\Omega$. Consider the bilinear form $A:H^1_0(\Omega)\times H^1_0(\Omega)\to\mathbb{R}$ defined by

$$A(w,v):=(\varepsilon\nabla w,\nabla v)+(\boldsymbol{\beta}\cdot\nabla w,v)+(\delta w,v),\quad \forall\, w,v\in H^1_0(\Omega),$$

and the linear functional $F:H^1_0(\Omega)\to\mathbb{R}$ defined by

$$F(v):=(f,v),\quad \forall\, v\in H^1_0(\Omega).$$

The variational formulation of (2.1) reads as: Finding $y\in H^1_0(\Omega)$ such that

$$A(y,v)=F(v),\quad \forall\, v\in H^1_0(\Omega). \tag{2.2}$$

The bilinear form A is coercive and bounded, and the variational problem (2.2) has a unique solution in view of the Lax-Milgram lemma.

2.2. Virtual element space

Let $\mathcal{T}_h$ be a family of decompositions of the domain Ω into non-overlapping polygonal elements whose boundaries are not self-intersecting. The maximum diameter of element E is denoted by h_E and $h=\sup_{E\in\mathcal{T}_h}h_E$. We further assume that ∂E denotes the edges of $E\in\mathcal{T}_h$. ∂E is made of a uniformly bounded number of line segments, which are either part of the boundary of Ω or shared with another element in the decomposition. $\mathcal{S}_h$ is the set of edges s of $\mathcal{T}_h$, which is subdivided into the set of boundary edges $\mathcal{S}_h^{bdry}:=\{s\in\mathcal{S}_h: s\subset\partial\Omega\}$ and the set of internal edges $\mathcal{S}_h^{int}:=\mathcal{S}_h\backslash\mathcal{S}_h^{bdry}$. The unit outward normal vector to ∂E and the length of edge s are denoted by $\boldsymbol{n}^E$ and h_s, respectively. We make the following assumption on the mesh for the theoretical analysis.

Assumption 2.1 (Mesh Regularity, [17]). *We assume the existence of a constant $\rho>0$ such that*

- *Every element E of $\mathcal{T}_h$ is star-shaped with respect to a disc of radius bigger or equal to ρh_E.*
- *For every element E of $\mathcal{T}_h$ and every side s of $E, h_s\geq\rho h_E$.*

For $m\in\mathbb{N}, q=1,\dots,\infty$, and any $E\in\mathcal{T}_h$, we introduce the following space:

$$W^m_q(\mathcal{T}_h):=\left\{v\in L^2(\Omega)\ \text{s.t.}\ v\,|_E\in W^m_q(E),\ \forall\, E\in\mathcal{T}_h\right\}$$

equipped with the broken norm and seminorm

$$\|v\|^q_{W^m_q(\mathcal{T}_h)}:=\sum_{E\in\mathcal{T}_h}\|v\|^q_{W^m_q(E)},\quad |v|^q_{W^m_q(\mathcal{T}_h)}:=\sum_{E\in\mathcal{T}_h}|v|^q_{W^m_q(E)},\quad \text{if}\quad 1\leq q<\infty,$$

$$\|v\|_{W^m_q(\mathcal{T}_h)}:=\max_{E\in\mathcal{T}_h}\|v\|_{W^m_q(E)},\quad |v|_{W^m_q(\mathcal{T}_h)}:=\max_{E\in\mathcal{T}_h}|v|_{W^m_q(E)},\quad \text{if}\quad q=\infty.$$

We set $W_2^m(\mathcal{T}_h) = H^m(\mathcal{T}_h)$ and $W_q^0(\mathcal{T}_h) = L^q(\mathcal{T}_h)$.

To define the virtual element space the following polynomial projections are defined [10]:

- The L^2-projection $\Pi_n^0 : L^2(E) \to \mathbb{P}_n(E)$, defined by

$$(q_n, v)_{0,E} = \left(q_n, \Pi_n^0 v\right)_{0,E}, \quad \forall\, v \in L^2(E), \quad q_n \in \mathbb{P}_n(E)$$

  with obvious extension for vector functions $\mathbf{\Pi}_n^0 : [L^2(E)]^2 \to [\mathbb{P}_n(E)]^2$.

- The H^1-projection $\Pi_{n+1}^\nabla : H^1(E) \to \mathbb{P}_{n+1}(E)$, defined by

$$(\nabla q_{n+1}, \nabla v)_{0,E} = \left(\nabla q_{n+1}, \nabla \Pi_{n+1}^\nabla v\right)_{0,E}, \quad \forall\, v \in H^1(E), \quad q_n \in \mathbb{P}_{n+1}(E)$$

  plus (to take care of the constant part of $\Pi_{n+1}^\nabla v$)

$$\frac{1}{|\partial E|} \int_{\partial E} \Pi_{n+1}^\nabla v ds = \frac{1}{|\partial E|} \int_{\partial E} v ds.$$

Then following [1] the local virtual space of order $k > 0$ is defined as follows:

$$V_{h,k}^E := \left\{ v_h \in \widetilde{V}_{h,k}^E : (v_h, p)_{0,E} = \left(\Pi_k^\nabla v_h, p\right)_{0,E},\ \forall\, p \in \mathbb{P}_k(E)/\mathbb{P}_{k-2}(E) \right\}, \quad \forall\, E \in \mathcal{T}_h,$$

where

$$\widetilde{V}_{h,k}^E = \left\{ v_h \in H^1(E) : v_h|_{\partial E} \in \mathbb{B}_k(\partial E),\ \Delta v_h \in \mathbb{P}_k(E) \right\},$$
$$\mathbb{B}_k(\partial E) = \left\{ v_h \in C^0(\partial E) : v_h|_s \in \mathbb{P}_k(s),\ \forall\, s \subset \partial E \right\}.$$

The global VEM space $V_{h,k}$ is defined by

$$V_{h,k} := \left\{ v_h \in H_0^1(\Omega) : v_h|_E \in V_{h,k}^E,\ \forall\, E \in \mathcal{T}_h \right\}.$$

For the selection of degrees of freedom and the more details about the practical aspects of the implementation of the VEM, we can refer to reference [1, 4, 18].

2.3. SUPG-VEM formulation

Set

$$V := \left\{ v \in H_0^1(\Omega) : v \in H^2(E),\ \forall\, E \in \mathcal{T}_h \right\}.$$

We define the bilinear form $A_{supg} : V \times H_0^1(\Omega) \to \mathbb{R}$

$$A_{supg}(w, v) := a(w, v) + b(w, v) + c(w, v) + d(w, v),$$

where

$$a(w, v) := \sum_{E \in \mathcal{T}_h} \left((\varepsilon \nabla w, \nabla v)_{0,E} + \tau_E (\boldsymbol{\beta} \cdot \nabla w, \boldsymbol{\beta} \cdot \nabla v)_{0,E}\right),$$
$$b(w, v) := \frac{1}{2} \left[(\boldsymbol{\beta} \cdot \nabla w, v) - (w, \boldsymbol{\beta} \cdot \nabla v)\right] = \frac{1}{2} \sum_{E \in \mathcal{T}_h} \left[(\boldsymbol{\beta} \cdot \nabla w, v)_{0,E} - (w, \boldsymbol{\beta} \cdot \nabla v)_{0,E}\right],$$
$$c(w, v) := \sum_{E \in \mathcal{T}_h} (\delta w, v + \tau_E \boldsymbol{\beta} \cdot \nabla v)_{0,E},$$
$$d(w, v) := \sum_{E \in \mathcal{T}_h} -\tau_E \varepsilon (\Delta w, \boldsymbol{\beta} \cdot \nabla v)_{0,E}.$$

Remark 2.1. Here the form $b(\cdot\,,\cdot)$ is rewritten as the skew-symmetric part, which is a useful step in order to preserve the coercivity of A_{supg} at the virtual discrete level, independently of the mesh size (see [7, 18]).

Furthermore, let $F_{supg} : H_0^1(\Omega) \to \mathbb{R}$ be the linear functional given by

$$F_{supg}(v) := (f, v) + \sum_{E\in\mathcal{T}_h} \tau_E(f, \boldsymbol{\beta}\cdot\nabla v)_{0,E}.$$

Then the SUPG variational formulation of problem (2.1) reads as: Finding $y \in V$ such that

$$A_{supg}(y, v) = F_{supg}(v), \quad \forall\, v \in H_0^1(\Omega). \tag{2.3}$$

The SUPG-stabilized virtual element approximation of (2.1) is to find $y_h \in V_{h,k}$ such that

$$A_{supg,h}(y_h, v_h) = F_{supg,h}(v_h), \quad v_h \in V_{h,k}. \tag{2.4}$$

Here $A_{supg,h}$ and $F_{supg,h}$ are defined as follows:

$$A_{supg,h}(w_h, v_h) := a_h(w_h, v_h) + b_h(w_h, v_h) + c_h(w_h, v_h) + d_h(w_h, v_h), \quad \forall\, w_h, v_h \in V_{h,k},$$

where

$$\begin{aligned}
a_h(w_h, v_h) &:= \sum_{E\in\mathcal{T}_h} a_h^E(w_h, v_h)\\
&= \sum_{E\in\mathcal{T}_h} \Big(\big(\varepsilon \boldsymbol{\Pi}_{k-1}^0 \nabla w_h, \boldsymbol{\Pi}_{k-1}^0 \nabla v_h\big)_{0,E} + \tau_E\big(\boldsymbol{\beta}\cdot \boldsymbol{\Pi}_{k-1}^0 \nabla w_h, \boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0\nabla v_h\big)_{0,E}\\
&\qquad + \big(\varepsilon + \tau_E \beta_E^2\big) S_a^E\big((I - \Pi_k^\nabla) w_h, (I - \Pi_k^\nabla) v_h\big)\Big),\\
b_h(w_h, v_h) &:= \sum_{E\in\mathcal{T}_h} b_h^E(w_h, v_h)\\
&= \sum_{E\in\mathcal{T}_h} \frac{1}{2}\Big(\big(\boldsymbol{\beta}\cdot\nabla\Pi_k^0 w_h, \Pi_k^0 v_h\big)_{0,E} - \big(\boldsymbol{\beta}\cdot\nabla\Pi_k^0 v_h, \Pi_k^0 w_h\big)_{0,E}\\
&\qquad + \int_{\partial E} \big(\boldsymbol{\beta}\cdot\boldsymbol{n}^E\big)\big(I - \Pi_k^0\big) w_h \Pi_k^0 v_h ds - \int_{\partial E}\big(\boldsymbol{\beta}\cdot\boldsymbol{n}^E\big)\big(I-\Pi_k^0\big) v_h \Pi_k^0 w_h ds\Big),\\
c_h(w_h, v_h) &:= \sum_{E\in\mathcal{T}_h} c_h^E(w_h, v_h)\\
&= \sum_{E\in\mathcal{T}_h} \Big(\big(\delta\Pi_k^0 w_h, \Pi_k^0 v_h + \tau_E \boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0\nabla v_h\big)_{0,E} + \delta S_b^E\big((I - \Pi_k^0) w_h, (I - \Pi_k^0) v_h\big)\Big),\\
d_h(w_h, v_h) &:= \sum_{E\in\mathcal{T}_h} d_h^E(w_h, v_h) = \sum_{E\in\mathcal{T}_h} -\tau_E\big(\nabla\cdot\big(\varepsilon\boldsymbol{\Pi}_{k-1}^0\nabla w_h\big), \boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0\nabla v_h\big)_{0,E}.
\end{aligned}$$

Here, for each element $E \in \mathcal{T}_h$,

$$\beta_E := \|\boldsymbol{\beta}(\boldsymbol{x})\|_{[L^\infty(E)]^2}.$$

Following [17, 18, 21] the local VEM stabilization term $S_a^E, S_b^E : V_{h,k}\times V_{h,k} \to \mathbb{R}$ in $a_h^E(w_h, v_h)$ and $c_h^E(w_h, v_h)$, respectively, must satisfy the following properties:

$$c_0\|\nabla v_h\|_{0,E}^2 \le S_a^E(v_h, v_h) \le c_1\|\nabla v_h\|_{0,E}^2, \quad \forall\, v_h \in \ker\Pi_k^\nabla, \tag{2.5}$$

$$c_2\|v_h\|_{0,E}^2 \le S_b^E(v_h, v_h) \le c_3\|v_h\|_{0,E}^2, \quad \forall\, v_h \in \ker\Pi_k^0, \tag{2.6}$$

where c_0, c_1, c_2 and c_3 are positive constants independent of h. According to [17,18,21] a possible choice for S_a^E and S_b^E is given by

$$S_a^E(v_h, w_h) = \sum_{r=1}^{N_E} \chi_r(w_h)\chi_r(v_h),$$
$$S_b^E(v_h, w_h) = h_E^2 \sum_{r=1}^{N_E} \chi_r(w_h)\chi_r(v_h),$$

where N_E is the number of degrees of freedom on the element E and χ_r is the operator that selects the r-th degree of freedom. The discrete right-hand side is defined as

$$F_{supg,h}(v_h) := \sum_{E\in\mathcal{T}_h} F_{supg,h}^E(v_h) = \sum_{E\in\mathcal{T}_h} \big(f, \Pi_k^0 v_h\big)_{0,E} + \sum_{E\in\mathcal{T}_h} \tau_E \big(f, \boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0 \nabla v_h\big)_{0,E}.$$

Following [11, 13, 28], the local SUPG parameter τ_E is chosen as

$$\tau_E := \min\left\{\frac{C_k^E h_E^2}{8\varepsilon}, \frac{h_E}{2\delta_0\beta_E}, \frac{C_\tau}{\delta}\right\} = \frac{h_E}{2\delta_0\beta_E} \tag{2.7}$$

in the convection dominated case. Here $C_\tau \in (0, 2)$ and $\delta_0 = \max\{k, C_{bub}^2\}$ are user-chosen constants, where C_{bub} is a constant defined below in Lemmas 3.5 and 3.6. C_k^E is the biggest constant number satisfying the following inverse inequality [11, 13]:

$$C_k^E h_E^2 \|\Delta v_h\|_{0,E}^2 \le \|\nabla v_h\|_{0,E}^2, \quad \forall\, v_h \in V_{h,k}^E. \tag{2.8}$$

According [13], the mesh Péclet number

$$P_{e_E} := \frac{C_k^E \beta_E h_E}{4\varepsilon} \gg 1$$

of element E is introduced and $K_{a_E} := 2\beta_E C_\tau/(h_E\delta)$ is the local Karlovitz number, i.e. the dimensionless parameter associated with each mesh element E. Then τ_E can be redefined as

$$\tau_E = \frac{h_E}{2\delta_0\beta_E} \min\{P_{e_E}\delta_0, 1, K_{a_E}\delta_0\}. \tag{2.9}$$

3. A Posteriori Error Analysis

In this section we first introduce some useful properties and assumptions. Then a global upper bound and a local lower bound for the a posteriori error are derived.

3.1. Some properties and assumptions

The usual local norm introduced in standard SUPG theory [24] is given by

$$\|v_h\|_{supg,E}^2 := \varepsilon\|\nabla v_h\|_{0,E}^2 + \tau_E\|\boldsymbol{\beta}\cdot\nabla v_h\|_{0,E}^2 + \left\|\delta^{\frac{1}{2}} v_h\right\|_{0,E}^2$$

for any $v_h \in H^1(E)$ and the global standard SUPG norm

$$\|v_h\|_{supg}^2 := \sum_{E\in\mathcal{T}_h} \|v_h\|_{supg,E}^2$$

for all $v_h \in H^1(\Omega)$. The following lemma gives the approximation properties of the projections $\Pi_n^0, \boldsymbol{\Pi}_n^0$ and Π_n^∇.

Lemma 3.1 ([5, 10]). *Given $E \in \mathcal{T}_h$, let φ and $\boldsymbol{\varphi}$ be sufficiently smooth scalar and vector valued functions, respectively. Then, it holds, for all $n \geq 0$,*

$$\begin{aligned}
&\left\|\varphi - \Pi_n^0 \varphi\right\|_{m,E} \leq C h_E^{r-m} |\varphi|_{r,E}, \quad 0 \leq m \leq r \leq n+1, \\
&\left\|\boldsymbol{\varphi} - \boldsymbol{\Pi}_n^0 \varphi\right\|_{m,E} \leq C h_E^{r-m} |\boldsymbol{\varphi}|_{r,E}, \quad 0 \leq m \leq r \leq n+1, \\
&\left\|\varphi - \Pi_n^\nabla \varphi\right\|_{m,E} \leq C h_E^{r-m} |\varphi|_{r,E}, \quad 0 \leq m \leq r \leq n+1, \quad r \geq 1,
\end{aligned}$$

where $C > 0$ only depends on the shape-regularity parameter ρ in Assumption 2.1 and k.

Lemma 3.2 ([30]). *Under Assumption 2.1, for every $v \in H^{1+r}(E)$ with $0 \leq r \leq k$, there exists $v_I \in V_{h,k}$ and a constant $C > 0$ such that*

$$\|v - v_I\|_{0,E} + h_E |v - v_I|_{1,E} \leq C h_E^{1+r} |v|_{1+r,E}.$$

Lemma 3.3 ([19]). *Under Assumption 2.1, the following inverse inequality holds:*

$$\|\nabla v_h\|_{0,E} \leq C h_E^{-1} \|v_h\|_{0,E}, \quad \forall\, v \in V_{h,k}^E,$$

where $C > 0$ is a constant independent of h.

The coercive proofs of $A_{supg,h}$ and A_{supg} are given below.

Lemma 3.4 ([7]). *Under Assumption 2.1 and in the case of convection dominated regime, i.e. τ_E satisfies (2.7), the bilinear form $A_{supg,h}$ satisfies for all $v_h \in V_{h,k}$ the coerciveness inequality*

$$A_{supg,h}(v_h, v_h) \geq C \|v_h\|_{supg}^2,$$

where C is a positive constant independent of h, τ_E, β_E and ε.

Proof. We introduce a local SUPG VEM norm

$$\begin{aligned}
\|v_h\|_{SUPG}^2 := &\, \varepsilon \|\nabla v\|_{0,E}^2 + \tau_E \left\|\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_{k-1}^0 \nabla v_h\right\|_{0,E}^2 + \tau_E \beta_E^2 \left\|\nabla\left(I - \Pi_k^\nabla\right) v_h\right\|_{0,E}^2 \\
&+ \left\|\delta^{\frac{1}{2}} \Pi_k^0 v_h\right\|_{0,E}^2 + \left\|\delta^{\frac{1}{2}} \left(I - \Pi_k^0\right) v_h\right\|_{0,E}^2.
\end{aligned}$$

Then, following [7, Proposition 5.1], we can prove that $A_{supg,h}(v_h, v_h) \geq \|v_h\|_{SUPG}$ and $\|\cdot\|_{supg}$ can be controlled by $\|\cdot\|_{SUPG}$. The coercivity can be obtained. □

Theorem 3.1. *Define $z := y - y_h, y \in V$ is the solution of (2.3) and $y_h \in V_{h,k}$ is SUPG VEM approximation computed by solving (2.4). Then the bilinear form A_{supg} satisfies*

$$A_{supg}(z, z) \geq C\left(\|z\|_{supg}^2 - \sum_{E \in \mathcal{T}_h} \tau_E \varepsilon^2 \|\Delta(y - y_I)\|_{0,E}^2 - \sum_{E \in \mathcal{T}_h} \tau_E \varepsilon^2 h_E^{-2} \|\nabla(y - y_I)\|_{0,E}^2\right),$$

where C is a positive constant independent of h, τ_E, β_E and ε, and $y_I \in V_{h,k}$ is the interpolant of y satisfying the bounds of Lemma 3.2.

Proof. By the definitions of $a(\cdot\,,\cdot), b(\cdot\,,\cdot)$ and $c(\cdot\,,\cdot)$, we have

$$\begin{aligned}
a(z, z) &= \varepsilon \|\nabla z\|_{0,\Omega}^2 + \sum_{E \in \mathcal{T}_h} \tau_E \|\boldsymbol{\beta} \cdot \nabla z\|_{0,E}^2, \\
b(z, z) &= 0, \\
c(z, z) &= \left\|\delta^{\frac{1}{2}} z\right\|_{0,\Omega}^2 + \sum_{E \in \mathcal{T}_h} \tau_E (\delta z, \boldsymbol{\beta} \cdot \nabla z)_{0,E}.
\end{aligned}$$

Using (2.7), the Cauchy-Schwarz and Young's inequalities we find that

$$\begin{aligned}
&\Big|\sum_{E\in\mathcal{T}_h}\tau_E(\delta z,\boldsymbol{\beta}\cdot\nabla z)_{0,E}\Big|\\
&\le \sum_{E\in\mathcal{T}_h}\tau_E\delta^{\frac12}\big\|\delta^{\frac12}z\big\|_{0,E}\|\boldsymbol{\beta}\cdot\nabla z\|_{0,E}\\
&\le \sum_{E\in\mathcal{T}_h}\Big(\frac{\delta\tau_E}{2}\big\|\delta^{\frac12}z\big\|_{0,E}^2+\frac{\tau_E}{2}\|\boldsymbol{\beta}\cdot\nabla z\|_{0,E}^2\Big)\\
&\le \sum_{E\in\mathcal{T}_h}\Big(\frac{C_\tau}{2}\big\|\delta^{\frac12}z\big\|_{0,E}^2+\frac{\tau_E}{2}\|\boldsymbol{\beta}\cdot\nabla z\|_{0,E}^2\Big),
\end{aligned}$$

which implies that

$$\sum_{E\in\mathcal{T}_h}\tau_E(\delta z,\boldsymbol{\beta}\cdot\nabla z)_{0,E}\ge -\frac{C_\tau}{2}\big\|\delta^{\frac12}z\big\|_{0,\Omega}^2-\sum_{E\in\mathcal{T}_h}\frac{\tau_E}{2}\|\boldsymbol{\beta}\cdot\nabla z\|_{0,E}^2.$$

For the last term $d(v,v)$, using the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon(\Delta z,\boldsymbol{\beta}\cdot\nabla z)_{0,E}\le\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\|\Delta z\|_{0,E}^2+\frac14\sum_{E\in\mathcal{T}_h}\tau_E\|\boldsymbol{\beta}\cdot\nabla z\|_{0,E}^2. \tag{3.1}$$

For the first term on the right-hand side of (3.1), adding and subtracting y_I yields

$$\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\|\Delta z\|_{0,E}^2\le 2\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\big(\|\Delta(y-y_I)\|_{0,E}^2+\|\Delta(y_I-y_h)\|_{0,E}^2\big).$$

From the inverse estimate (2.8), adding and subtracting y, and (2.7), we infer

$$\begin{aligned}
&\sum_{E\in\mathcal{T}_h}2\tau_E\varepsilon^2\|\Delta(y_I-y_h)\|_{0,E}^2\\
&\le\sum_{E\in\mathcal{T}_h}2\tau_E\varepsilon^2\big(C_k^E\big)^{-1}h_E^{-2}\|\nabla(y_I-y_h)\|_{0,E}^2\\
&\le 4\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\big(C_k^E\big)^{-1}h_E^{-2}\big(\|\nabla z\|_{0,E}^2+\|\nabla(y-y_I)\|_{0,E}^2\big)\\
&\le\frac12\varepsilon\|\nabla z\|_{0,\Omega}^2+4\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\big(C_k^E\big)^{-1}h_E^{-2}\|\nabla(y-y_I)\|_{0,E}^2.
\end{aligned}$$

We further get

$$\begin{aligned}
&-\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon(\Delta z,\boldsymbol{\beta}\cdot\nabla z)_{0,E}\\
&\ge -2\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\|\Delta(y-y_I)\|_{0,E}^2-\frac12\varepsilon\|\nabla z\|_{0,\Omega}^2\\
&\quad -4\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\big(C_k^E\big)^{-1}h_E^{-2}\|\nabla(y-y_I)\|_{0,E}^2-\frac14\sum_{E\in\mathcal{T}_h}\tau_E\|\boldsymbol{\beta}\cdot\nabla z\|_{0,E}^2.
\end{aligned}$$

Collecting the previous bounds, we obtain

$$A_{supg}(v,v)\ge\frac12\varepsilon\|\nabla v\|_{0,\Omega}^2+\frac14\sum_{E\in\mathcal{T}_h}\tau_E\|\boldsymbol{\beta}\cdot\nabla v\|_{0,E}^2+\frac{2-C_\tau}{2}\big\|\delta^{\frac12}v\big\|_{0,\Omega}^2$$

$$
\begin{aligned}
&- 2 \sum_{E\in\mathcal{T}_h} \tau_E \varepsilon^2 \|\Delta(y-y_I)\|_{0,E}^2 - 4 \sum_{E\in\mathcal{T}_h} \tau_E \varepsilon^2 \big(C_k^E\big)^{-1} h_E^{-2} \|\nabla(y-y_I)\|_{0,E}^2 \\
\geq & \min\left\{\frac{2-C_\tau}{2}, \frac{1}{4}\right\} \|v\|_{supg}^2 - 2 \sum_{E\in\mathcal{T}_h} \tau_E \varepsilon^2 \|\Delta(y-y_I)\|_{0,E}^2 \\
&- 4 \sum_{E\in\mathcal{T}_h} \tau_E \varepsilon^2 \big(C_k^E\big)^{-1} h_E^{-2} \|\nabla(y-y_I)\|_{0,E}^2.
\end{aligned}
$$

The theorem result can be obtained. □

For the additional terms in Theorem 3.1, we have the following analysis.

Theorem 3.2. *We have the following approximation property for all $y \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$, $0 < r \leq k$:*

$$
\sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \|\Delta(y-y_I)\|_{0,E}^2 \leq C \sum_{E\in\mathcal{T}_h} P_{e_E}^{-2} \beta_E h_E^{2r+1} \|y\|_{r+1,E}^2,
$$

where C is a positive constant independent of h, τ_E, ε and β_E.

Proof. The triangle inequality, Lemma 3.1, and the inverse inequality (2.8) yield

$$
\begin{aligned}
\varepsilon \tau_E^{\frac{1}{2}} \|\Delta(y-y_I)\|_{0,E} &= \varepsilon \tau_E^{\frac{1}{2}} \big\|\Delta\big(y - \Pi_k^0 y\big)\big\|_{0,E} + \varepsilon \tau_E^{\frac{1}{2}} \big\|\Delta\big(\Pi_k^0 y - y_I\big)\big\|_{0,E} \\
&\leq \varepsilon \tau_E^{\frac{1}{2}} \big\|y - \Pi_k^0 y\big\|_{2,E} + \varepsilon \tau_E^{\frac{1}{2}} \big(C_k^E\big)^{-\frac{1}{2}} h_E^{-1} \big\|\nabla\big(\Pi_k^0 y - y_I\big)\big\|_{0,E} \\
&\leq C \varepsilon \tau_E^{\frac{1}{2}} h_E^{r-1} \|y\|_{r+1,E} + \varepsilon \tau_E^{\frac{1}{2}} \big(C_k^E\big)^{-\frac{1}{2}} h_E^{-1} \big\|\Pi_k^0 y - y_I\big\|_{1,E}.
\end{aligned}
$$

Using (2.7), we further have

$$
\begin{aligned}
&\sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \|\Delta(y-y_I)\|_{0,E}^2 \\
\leq & C \sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E h_E^{2r-2} \|y\|_{r+1,E}^2 + \sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \big(C_k^E\big)^{-1} h_E^{-2} \big\|\Pi_k^0 y - y_I\big\|_{1,E}^2 \\
\leq & C \sum_{E\in\mathcal{T}_h} \beta_E h_E^{2r+1} P_{e_E}^{-2} \|y\|_{r+1,E}^2 + \sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \big(C_k^E\big)^{-1} h_E^{-2} \big\|\Pi_k^0 y - y_I\big\|_{1,E}^2. \qquad (3.2)
\end{aligned}
$$

The last term can be estimated using the Lemmas 3.1, 3.2 and (2.7)

$$
\begin{aligned}
&\sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \big(C_k^E\big)^{-1} h_E^{-2} \big\|\Pi_k^0 y - y_I\big\|_{1,E}^2 \\
\leq & 2 \sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \big(C_k^E\big)^{-1} h_E^{-2} \big\|\Pi_k^0 y - y\big\|_{1,E}^2 + 2 \sum_{E\in\mathcal{T}_h} \varepsilon^2 \tau_E \big(C_k^E\big)^{-1} h_E^{-2} \|y - y_I\|_{1,E}^2 \\
\leq & C \sum_{E\in\mathcal{T}_h} P_{e_E}^{-2} \beta_E h_E^{2r+1} \|y\|_{r+1,E}^2.
\end{aligned}
$$

Substituting this estimate into (3.2) leads to the theorem result. □

To derive a robust posteriori error estimate we need to introduce the following assumption, which is also used in the SUPG finite element framework in [28].

Assumption 3.1. *Let* $y \in H_0^1(\Omega) \cap H^{r+1}(\mathcal{T}_h), f \in H^r(\mathcal{T}_h), \boldsymbol{\beta} \in [W_\infty^r(\mathcal{T}_h)]^2, 0 < r \leq k,$ *and* $y_I \in V_{h,k}$ *be the interpolant of* y *satisfying the bounds of Lemma* 3.2*. The interpolation error* $y - y_I$ *are assumed to be bounded by the error* $y - y_h$

$$\sum_{E\in\mathcal{T}_h} \tau_E^{-1}\|y - y_I\|_{0,E}^2 \leq C\|y - y_h\|_{supg}^2,$$
$$\sum_{E\in\mathcal{T}_h} \tau_E\|\boldsymbol{\beta}\cdot\nabla(y - y_I)\|_{0,E}^2 \leq C\|y - y_h\|_{supg}^2,$$
$$\sum_{E\in\mathcal{T}_h} \left(\beta_{\omega_E} \sum_{s\subset\partial E} \|y - y_I\|_{0,s}^2\right) \leq C\|y - y_h\|_{supg}^2,$$

where $\omega_E := \{E' \in \mathcal{T}_h : \mu_1(\partial E' \cap \partial E) \neq 0\}$ *is the patch made up of the element* E *and its neighbours, and* μ_1 *is the one-dimensional measure.* $\beta_{\omega_E} = \|\boldsymbol{\beta}\|_{[L^\infty(\omega_E)]^2}$*.* $C > 0$ *is a constant independent of the mesh size* h, τ_E *and* ε*.*

Remark 3.1 (Discussion of Assumption 3.1). Assumption 3.1 is only needed in the convection dominated case. In [7], under the above data assumption a robustness analysis of the SUPG-stabilized virtual elements for convection-diffusion problem is discussed and the optimal order of convergence is obtained, i.e.

$$\|y - y_h\|_{supg}^2 = \mathcal{O}(h^{2r+1}). \tag{3.3}$$

Through a similar analysis, this result is also true for the convection-diffusion-reaction problem in our paper. Using the optimal choice $\tau_E = \mathcal{O}(h_E)$ of the stabilization parameter, as given e.g. by (2.7), one gets from the interpolation estimate Lemma 3.2

$$\sum_{E\in\mathcal{T}_h} \tau_E^{-1}\|y - y_I\|_{0,E}^2 = \mathcal{O}(h^{2r+1}),$$
$$\sum_{E\in\mathcal{T}_h} \tau_E\|\boldsymbol{\beta}\cdot\nabla(y - y_I)\|_{0,E}^2 \leq \|\boldsymbol{\beta}\|_{[L^\infty(\mathcal{T}_h)]^2}^2 \sum_{E\in\mathcal{T}_h} \tau_E\|\nabla(y - y_I)\|_{0,E}^2 = \mathcal{O}(h^{2r+1}).$$

By the trace inequality [14], we have

$$\|w\|_{0,s}^2 \leq C\big(h_E^{-1}\|w\|_{0,E}^2 + h_E\|\nabla w\|_{0,E}^2\big)$$

for $w \in H^1(E)$, and Lemma 3.2, we have

$$\begin{aligned}&\sum_{E\in\mathcal{T}_h} \left(\beta_{\omega_E} \sum_{s\subset\partial E} \|y - y_I\|_{0,s}^2\right)\\ &\leq C\|\boldsymbol{\beta}\|_{[L^\infty(\mathcal{T}_h)]^2} \sum_{E\in\mathcal{T}_h} \big(h_E^{-1}\|y - y_I\|_{0,E}^2 + h_E\|\nabla(y - y_I)\|_{0,E}^2\big) = \mathcal{O}(h^{2r+1}).\end{aligned}$$

Hence, all terms in Assumption 3.1 are of the same order with respect to h, i.e. $\mathcal{O}(h^{2r+1})$. When there exist sharp boundary or inner layers in the solution, the numerical results obtained with the SUPG method are generally not perfect because there are often spurious oscillations in a vicinity of layers. In contrast, the interpolation y_I of y is exact at the mesh nodes, which can ensures that Assumption 3.1 holds since the there is no spurious oscillations in the interpolant justifies.

The following results which establish standard estimates for bubble functions will be useful in what follows.

Lemma 3.5 (Interior Bubble Functions, [17]). *Let $E \in \mathcal{T}_h$ and Ψ_E be the corresponding bubble function. There exists a constant C_{bub}, independent of h_E such that for all $q \in \mathbb{P}_k(E)$,*

$$C_{bub}^{-1}\|q\|_{0,E}^2 \le \int_E \Psi_E q^2 dx \le C_{bub}\|q\|_{0,E}^2,$$
$$C_{bub}^{-1}\|q\|_{0,E} \le \|\Psi_E q\|_{0,E} + h_E\|\nabla(\Psi_E q)\|_{0,E} \le C_{bub}\|q\|_{0,E}.$$

Lemma 3.6 (Edge Bubble Functions, [17]). *For $E \in \mathcal{T}_h$, let $s \subset \partial E$ be a mesh interface and Ψ_s be the corresponding interface bubble function. There exists a constant C_{bub}, independent of h_E such that for all $q \in \mathbb{P}_k(s)$,*

$$C_{bub}^{-1}\|q\|_{0,s}^2 \le \int_s \Psi_s q^2 ds \le C_{bub}\|q\|_{0,s}^2,$$
$$h_E^{-\frac{1}{2}}\|\Psi_s q\|_{0,E} + h_E^{\frac{1}{2}}\|\nabla(\Psi_s q)\|_{0,E} \le C_{bub}\|q\|_{0,s}.$$

3.2. Upper bound

To illustrate the impacts of data oscillation, piecewise constant approximation of $\boldsymbol{\beta}$ is introduced: $\boldsymbol{\beta}_h \approx \boldsymbol{\beta}$. For quantities which may be discontinuous across the mesh skeleton, the jump operator $[\![\cdot]\!]$ across a mesh edge $s \in \mathcal{S}_h$ is defined as follows. If $s \in \mathcal{S}_h^{int}$, then there exist E^+ and E^- such that $s \subset \partial E^+ \cap \partial E^-$. Denote by $\boldsymbol{v}^\pm$ the trace of the vector-valued function $\boldsymbol{v}|_{E^\pm}$ on s from within $E^\pm$ and by $\boldsymbol{n}_s^\pm$ the unit outward normal on s from $E^\pm$. Then,

$$[\![\boldsymbol{v}]\!] := \boldsymbol{v}^+ \cdot \boldsymbol{n}_s^+ + \boldsymbol{v}^- \cdot \boldsymbol{n}_s^-.$$

If $s \in \mathcal{S}_h^{bdry}$, then $[\![v]\!] := \boldsymbol{v} \cdot \boldsymbol{n}_s$, with $\boldsymbol{v}$ representing the trace of $\boldsymbol{v}$ from within the element E having s as an edge and $\boldsymbol{n}_s$ is the unit outward normal on s from E.

Recalling that $z = y - y_h$. Then, we have the residual equation

$$\begin{aligned}
A_{supg}(z,v) &= (f,v) + \sum_{E\in\mathcal{T}_h}(f,\tau_E\boldsymbol{\beta}\cdot\nabla v)_{0,E} - A_{supg}(y_h,\lambda) - A_{supg}(y_h, v-\lambda)\\
&= \sum_{E\in\mathcal{T}_h}(f, v+\tau_E\boldsymbol{\beta}\cdot\nabla v)_{0,E} - \sum_{E\in\mathcal{T}_h}\big(f, \Pi_k^0\lambda + \tau_E\boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0\nabla\lambda\big)_{0,E}\\
&\quad + A_{supg,h}(y_h,\lambda) - A_{supg}(y_h,\lambda) - A_{supg}(y_h,v-\lambda)\\
&= \sum_{E\in\mathcal{T}_h}\big(f-\Pi_k^0 f,\lambda\big)_{0,E} + (f, v-\lambda) + \sum_{E\in\mathcal{T}_h}\tau_E\big(f\boldsymbol{\beta} - \boldsymbol{\Pi}_{k-1}^0(f\boldsymbol{\beta}),\nabla\lambda\big)_{0,E}\\
&\quad + \sum_{E\in\mathcal{T}_h}\tau_E(f\boldsymbol{\beta}, \nabla v - \nabla\lambda)_{0,E} + A_{supg,h}(y_h,\lambda)\\
&\quad - A_{supg}(y_h,\lambda) - A_{supg}(y_h,v-\lambda)
\end{aligned}\tag{3.4}$$

for any $\lambda \in V_{h,k}, v \in H_0^1(\Omega)$.

In order to obtain a computable error bound, we estimate each term on the right-hand side of the above residual equation separately. By applying integration by parts to last term, we have

$$A_{supg}(y_h, v-\lambda) = \sum_{E\in\mathcal{T}_h}\big(-\nabla\cdot\varepsilon\boldsymbol{\Pi}_{k-1}^0\nabla y_h + \boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0\nabla y_h + \delta\Pi_k^0 y_h, v-\lambda\big)_{0,E} \tag{3.5}$$

$$
\begin{aligned}
&+\sum_{s\in\mathcal{S}_h}\int_s \varepsilon[\![\mathbf{\Pi}^0_{k-1}\nabla y_h]\!](v-\lambda)ds+\sum_{E\in\mathcal{T}_h}\big(\varepsilon(I-\mathbf{\Pi}^0_{k-1})\nabla y_h,\nabla(v-\lambda)\big)_{0,E}\\
&+\sum_{E\in\mathcal{T}_h}\big(\boldsymbol{\beta}\cdot(I-\mathbf{\Pi}^0_{k-1})\nabla y_h+\delta(I-\Pi^0_k)y_h,v-\lambda\big)_{0,E}\\
&+\sum_{E\in\mathcal{T}_h}\big(-\nabla\cdot\varepsilon\mathbf{\Pi}^0_{k-1}\nabla y_h+\boldsymbol{\beta}\cdot\mathbf{\Pi}^0_{k-1}\nabla y_h+\delta\Pi^0_k y_h,\tau_E\boldsymbol{\beta}\cdot\nabla(v-\lambda)\big)_{0,E}\\
&+\sum_{E\in\mathcal{T}_h}\big(\nabla\cdot\varepsilon(\mathbf{\Pi}^0_{k-1}-I)\nabla y_h+\boldsymbol{\beta}\cdot(I-\mathbf{\Pi}^0_{k-1})\nabla y_h+\delta(I-\Pi^0_k)y_h,\tau_E\boldsymbol{\beta}\cdot\nabla(v-\lambda)\big)_{0,E}.
\end{aligned}
$$

Using (3.5), (3.4) can be rewritten as

$$
\begin{aligned}
A_{supg}(z,v)=&\sum_{E\in\mathcal{T}_h}\Big(\big(R_E,v-\lambda+\tau_E\boldsymbol{\beta}\cdot\nabla(v-\lambda)\big)_{0,E}+\big(\theta_E,v-\lambda+\tau_E\boldsymbol{\beta}\cdot\nabla(v-\lambda)\big)_{0,E}\\
&\qquad+B^E_1(y_h,\lambda-v)+B^E_2(y_h,\lambda-v)\Big)\\
&+\sum_{s\in\mathcal{S}_h}(J_s,\lambda-v)_{0,s}+\sum_{E\in\mathcal{T}_h}\Big(\big(f-\Pi^0_k f,\lambda\big)_{0,E}+\tau_E\big(f\boldsymbol{\beta}-\mathbf{\Pi}^0_{k-1}(f\boldsymbol{\beta}),\nabla\lambda\big)_{0,E}\Big)\\
&+A_{supg,h}(y_h,\lambda)-A_{supg}(y_h,\lambda)
\end{aligned}
\tag{3.6}
$$

for any $\lambda\in V_{h,k}, v\in H^1_0(\Omega)$, where

$$
\begin{aligned}
&R_E:=\big(\Pi^0_k f+\nabla\cdot\varepsilon\mathbf{\Pi}^0_{k-1}\nabla y_h-\boldsymbol{\beta}_h\cdot\mathbf{\Pi}^0_{k-1}\nabla y_h-\delta\Pi^0_k y_h\big)|_E,\\
&J_s:=\varepsilon[\![\mathbf{\Pi}^0_{k-1}\nabla y_h]\!]|_s,\\
&\theta_E:=\big(f-\Pi^0_k f-(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\cdot\mathbf{\Pi}^0_{k-1}\nabla y_h\big)|_E
\end{aligned}
$$

are the element and edge residuals, and the element data oscillation terms, respectively

$$
\begin{aligned}
B^E_1(y_h,\lambda-v):=&\big(\varepsilon\big(I-\mathbf{\Pi}^0_{k-1}\big)\nabla y_h,\nabla(\lambda-v)\big)_{0,E}\\
&+\big(\boldsymbol{\beta}\cdot\big(I-\mathbf{\Pi}^0_{k-1}\big)\nabla y_h+\delta\big(I-\Pi^0_k\big)y_h,\lambda-v\big)_{0,E},\\
B^E_2(y_h,\lambda-v):=&\big(\nabla\cdot\varepsilon\big(\mathbf{\Pi}^0_{k-1}-I\big)\nabla y_h+\boldsymbol{\beta}\cdot\big(I-\mathbf{\Pi}^0_{k-1}\big)\nabla y_h\\
&+\delta\big(I-\Pi^0_k\big)y_h,\tau_E\boldsymbol{\beta}\cdot\nabla(\lambda-v)\big)_{0,E}
\end{aligned}
$$

are the virtual residuals.

Theorem 3.3 (Upper Bound). *Let $y_h\in V_{h,k}$ and $y\in V$ be the solutions to problems* (2.4) *and* (2.3)*, respectively. Then under Assumptions* 2.1 *and* 3.1*, there exists a constant C, independent of the mesh size h, the SUPG parameter τ_E and the diffusive coefficient ε such that*

$$
\begin{aligned}
\|y-y_h\|^2_{supg}\le C\sum_{E\in\mathcal{T}_h}&\big(\eta^E+\Theta^E+\Xi^E+\Psi^E+\tau_E\varepsilon^2\|\Delta(y-y_I)\|^2_{0,E}\\
&+\tau_E\varepsilon^2h_E^{-2}\|\nabla(y-y_I)\|^2_{0,E}\big),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
&\eta^E:=\tau_E\|R_E\|^2_{0,E}+\beta^{-1}_{\omega_E}\sum_{s\subset\partial E}\|J_s\|^2_{0,s},\\
&\Theta^E:=\tau_E\|\theta_E\|^2_{0,E}+\big(\delta^{-1}+\tau_E\big)\big\|f-\Pi^0_k f\big\|^2_{0,E}+\beta_E^{-2}\big(\tau_E+\delta^{-1}\big)\big\|f\boldsymbol{\beta}-\mathbf{\Pi}^0_{k-1}(f\boldsymbol{\beta})\big\|^2_{0,E},\\
&\Xi^E:=\delta^{-1}\beta_E^2S^E_a\big(\big(I-\Pi^\nabla_k\big)y_h,\big(I-\Pi^\nabla_k\big)y_h\big)+\delta S^E_b\big(\big(I-\Pi^0_k\big)y_h,\big(I-\Pi^0_k\big)y_h\big),
\end{aligned}
$$

and Ψ^E *encompasses the virtual inconsistency terms, defined as the sum of*

$$\begin{aligned}&\beta_E^{-2}\big(\tau_E+\delta^{-1}\big)\big\|\big(I-\mathbf{\Pi}_{k-1}^0\big)\boldsymbol{\beta}\boldsymbol{\beta}^T\mathbf{\Pi}_{k-1}^0\nabla y_h\big\|_{0,E}^2,\\&\big(\tau_E+\delta^{-1}\big)\big\|\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0 y_h\big\|_{0,E}^2,\\&\beta_E^{-2}\big(\tau_E\delta^2+\delta\big)\big\|\big(I-\mathbf{\Pi}_{k-1}^0\big)\Pi_k^0 y_h\boldsymbol{\beta}\big\|_{0,E}^2,\\&\tau_E^2\varepsilon\big\|\big(I-\mathbf{\Pi}_{k-1}^0\big)\big(\nabla\cdot\mathbf{\Pi}_{k-1}^0\nabla y_h\big)\boldsymbol{\beta}\big\|_{0,E}^2.\end{aligned}$$

Proof. Let $z_I\in V_{h,k}$ be the interpolant of z that satisfies Lemma 3.2. We have

$$z-z_I=y-y_h-(y_I-y_h)=y-y_I.$$

Setting $v=z$ and $\lambda=z_I$ in (3.6), from Theorem 3.1 we have

$$\begin{aligned}C\|z\|_{supg}^2&\le A_{supg}(z,z)+\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\|\Delta(y-y_I)\|_{0,E}^2+\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2h_E^{-2}\|\nabla(y-y_I)\|_{0,E}^2,\\&=\sum_{E\in\mathcal{T}_h}\Big(\big(R_E,z-z_I+\tau_E\boldsymbol{\beta}\cdot\nabla(z-z_I)\big)_{0,E}+\big(\theta_E,z-z_I+\tau_E\boldsymbol{\beta}\cdot\nabla(z-z_I)\big)_{0,E}\\&\qquad+B_1^E(y_h,z_I-z)+B_2^E(y_h,z_I-z)+\big(A_{supg,h}^E(y_h,z_I)-A_{supg}^E(y_h,z_I)\big)\Big)\\&\quad+\sum_{s\in\mathcal{S}_h}(J_s,z_I-z)_{0,s}+\sum_{E\in\mathcal{T}_h}\Big(\big(f-\Pi_k^0f,z_I\big)_{0,E}+\tau_E\big(f\boldsymbol{\beta}-\mathbf{\Pi}_{k-1}^0(f\boldsymbol{\beta}),\nabla z_I\big)_{0,E}\Big)\\&\quad+\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\|\Delta(y-y_I)\|_{0,E}^2+\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2h_E^{-2}\|\nabla(y-y_I)\|_{0,E}^2,\\&=:\sum_{i=1}^8M_i+\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2\|\Delta(y-y_I)\|_{0,E}^2+\sum_{E\in\mathcal{T}_h}\tau_E\varepsilon^2h_E^{-2}\|\nabla(y-y_I)\|_{0,E}^2.\end{aligned}$$

By the Cauchy-Schwarz inequality and Assumption 3.1 we have

$$\begin{aligned}M_1&\le\sum_{E\in\mathcal{T}_h}\|R_E\|_{0,E}\big(\|y-y_I\|_{0,E}+\tau_E\|\boldsymbol{\beta}\cdot\nabla(y-y_I)\|_{0,E}\big),\\&\le C\Big(\sum_{E\in\mathcal{T}_h}\tau_E\|R_E\|_{0,E}^2\Big)^{\frac12}\Big(\sum_{E\in\mathcal{T}_h}\big(\tau_E^{-1}\|y-y_I\|_{0,E}^2+\tau_E\|\boldsymbol{\beta}\cdot\nabla(y-y_I)\|_{0,E}^2\big)\Big)^{\frac12}\\&\le C\Big(\sum_{E\in\mathcal{T}_h}\tau_E\|R_E\|_{0,E}^2\Big)^{\frac12}\|z\|_{supg}.\end{aligned}$$

Similarly, we have the estimate

$$M_2\le C\Big(\sum_{E\in\mathcal{T}_h}\tau_E\|\theta_E\|_{0,E}^2\Big)^{\frac12}\|z\|_{supg}.$$

Note that

$$\begin{aligned}&\big\|\nabla v_h-\mathbf{\Pi}_{k-1}^0\nabla v_h\big\|_{0,E}=\big\|\big(I-\mathbf{\Pi}_{k-1}^0\big)\nabla\big(I-\Pi_k^\nabla\big)v_h\big\|_{0,E}\le\big\|\nabla\big(I-\Pi_k^\nabla\big)v_h\big\|_{0,E},\\&\big\|\big(I-\Pi_k^0\big)v_h\big\|_{0,E}=\big\|\big(I-\Pi_k^0\big)\big(I-\Pi_k^\nabla\big)v_h\big\|_{0,E}\le\big\|\big(I-\Pi_k^\nabla\big)v_h\big\|_{0,E}.\end{aligned}\tag{3.8}$$

By Assumption 3.1, (2.5), (2.7), (3.8) and Lemma 3.2, we can bound M_3 as follows:

$$
\begin{aligned}
M_3 \le & \sum_{E\in\mathcal{T}_h} \Big(\varepsilon\big\|\nabla\big(I-\Pi_k^{\nabla}\big)y_h\big\|_{0,E}\|\nabla(z_I-z)\|_{0,E} + \beta_E\big\|\nabla\big(I-\Pi_k^{\nabla}\big)y_h\big\|_{0,E}\|y_I-y\|_{0,E} \\
& \qquad + \delta\big\|\big(I-\Pi_k^0\big)y_h\big\|_{0,E}\|y_I-y\|_{0,E}\Big) \\
\le & \Big(\sum_{E\in\mathcal{T}_h}\varepsilon^2\big\|\nabla\big(I-\Pi_k^{\nabla}\big)y_h\big\|_{0,E}^2\Big)^{\frac{1}{2}}\|\nabla z\|_{0,\Omega} + \Big(\sum_{E\in\mathcal{T}_h}\beta_E^2\tau_E\big\|\nabla\big(I-\Pi_k^{\nabla}\big)y_h\big\|_{0,E}^2\Big)^{\frac{1}{2}} \\
& \times\Big(\sum_{E\in\mathcal{T}_h}\tau_E^{-1}\|y_I-y\|_{0,E}^2\Big)^{\frac{1}{2}} + \Big(\sum_{E\in\mathcal{T}_h}\delta^2\tau_E\big\|\big(I-\Pi_k^0\big)y_h\big\|_{0,E}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\tau_E^{-1}\|y_I-y\|_{0,E}^2\Big)^{\frac{1}{2}} \\
\le & C\|z\|_{supg}\Bigg(\Big(\sum_{E\in\mathcal{T}_h}(\varepsilon+\beta_E^2\tau_E)S_a^E\big(\big(I-\Pi_k^{\nabla}\big)y_h,\big(I-\Pi_k^{\nabla}\big)y_h\big)\Big)^{\frac{1}{2}} \\
& \qquad + \Big(\sum_{E\in\mathcal{T}_h}\delta^2\tau_E S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)y_h\big)\Big)^{\frac{1}{2}}\Bigg) \\
\le & C\|z\|_{supg}\Bigg(\sum_{E\in\mathcal{T}_h}\Big(\big(\varepsilon+\beta_E^2\tau_E\big)S_a^E\big(\big(I-\Pi_k^{\nabla}\big)y_h,\big(I-\Pi_k^{\nabla}\big)y_h\big) \\
& \qquad + \delta^2\tau_E S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)y_h\big)\Big)\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Using the same computations and inverse inequality (2.8), we can also infer

$$
\begin{aligned}
M_4 \le C\|z\|_{supg}\Bigg(\sum_{E\in\mathcal{T}_h}\Big(&\big(\varepsilon+\beta_E^2\tau_E\big)S_a^E\big(\big(I-\Pi_k^{\nabla}\big)y_h,\big(I-\Pi_k^{\nabla}\big)y_h\big) \\
& + \delta^2\tau_E S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)y_h\big)\Big)\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Since the estimation of M_5 is cumbersome, we do this work at the end. Applying the Cauchy-Schwarz inequality and Assumption 3.1 one obtains for M_6 that

$$
\begin{aligned}
M_6 &\le \sum_{s\in\mathcal{S}_h}\|J_s\|_{0,s}\|z_I-z\|_{0,s} \le \sum_{E\in\mathcal{T}_h}\Big(\sum_{s\subset\partial E}\|J_s\|_{0,s}\|z_I-z\|_{0,s}\Big) \\
&\le \sum_{E\in\mathcal{T}_h}\Bigg(\Big(\sum_{s\subset\partial E}\|J_s\|_{0,s}^2\Big)^{\frac{1}{2}}\Big(\sum_{s\subset\partial E}\|y_I-y\|_{0,s}^2\Big)^{\frac{1}{2}}\Bigg) \\
&\le \Big(\sum_{E\in\mathcal{T}_h}(\beta_{\omega_E}^{-1}\sum_{s\subset\partial E}\|J_s\|_{0,s}^2)\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{T}_h}\Big(\beta_{\omega_E}\sum_{s\subset\partial E}\|y_I-y\|_{0,s}^2\Big)\Big)^{\frac{1}{2}} \\
&\le C\Big(\sum_{E\in\mathcal{T}_h}\Big(\beta_{\omega_E}^{-1}\sum_{s\subset\partial E}\|J_s\|_{0,s}^2\Big)\Big)^{\frac{1}{2}}\|z\|_{supg}.
\end{aligned}
$$

We use Lemma 3.1, Assumption 3.1 and (2.7) to find that

$$
\begin{aligned}
M_7 &\le C\sum_{E\in\mathcal{T}_h}\big\|f-\Pi_k^0 f\big\|_{0,E}\|z_I\|_{0,E} \\
&\le C\Big(\sum_{E\in\mathcal{T}_h}\tau_E^{\frac{1}{2}}\big\|f-\Pi_k^0 f\big\|_{0,E}\tau_E^{-\frac{1}{2}}\|y_I-y\|_{0,E} + \sum_{E\in\mathcal{T}_h}\big\|f-\Pi_k^0 f\big\|_{0,E}\|y-y_h\|_{0,E}\Big)
\end{aligned}
$$

$$\leq C\|z\|_{supg}\bigg(\Big(\sum_{E\in\mathcal{T}_h}\tau_E\big\|f-\Pi_k^0 f\big\|_{0,E}^2\Big)^{\frac{1}{2}}+\Big(\sum_{E\in\mathcal{T}_h}\delta^{-1}\big\|f-\Pi_k^0 f\big\|_{0,E}^2\Big)^{\frac{1}{2}}\bigg)$$
$$\leq C\|z\|_{supg}\Big(\sum_{E\in\mathcal{T}_h}\big(\tau_E+\delta^{-1}\big)\big\|f-\Pi_k^0 f\big\|_{0,E}^2\Big)^{\frac{1}{2}}.$$

Employing Lemma 3.3 and Assumption 3.1, we obtain

$$M_8\leq\sum_{E\in\mathcal{T}_h}\tau_E\big\|f\boldsymbol{\beta}-\boldsymbol{\Pi}_{k-1}^0(f\boldsymbol{\beta})\big\|_{0,E}\|\nabla z_I\|_{0,E}$$
$$\leq C\sum_{E\in\mathcal{T}_h}\tau_E\big\|f\boldsymbol{\beta}-\boldsymbol{\Pi}_{k-1}^0(f\boldsymbol{\beta})\big\|_{0,E}h_E^{-1}\big(\|y_I-y\|_{0,E}+\|y-y_h\|_{0,E}\big)$$
$$\leq C\bigg(\Big(\sum_{E\in\mathcal{T}_h}\tau_E\beta_E^{-2}\big\|f\boldsymbol{\beta}-\boldsymbol{\Pi}_{k-1}^0(f\boldsymbol{\beta})\big\|_{0,E}^2\Big)^{\frac{1}{2}}\|y-y_h\|_{supg}$$
$$+\Big(\sum_{E\in\mathcal{T}_h}\beta_E^{-2}\delta^{-1}\big\|f\boldsymbol{\beta}-\boldsymbol{\Pi}_{k-1}^0(f\boldsymbol{\beta})\big\|_{0,E}^2\Big)^{\frac{1}{2}}\|y-y_h\|_{supg}\bigg)$$
$$\leq C\|z\|_{supg}\Big(\sum_{E\in\mathcal{T}_h}\beta_E^{-2}\big(\tau_E+\delta^{-1}\big)\big\|f\boldsymbol{\beta}-\boldsymbol{\Pi}_{k-1}^0(f\boldsymbol{\beta})\big\|_{0,E}^2\Big)^{\frac{1}{2}}.$$

Finally, we turn to the estimate of M_5, i.e. $A_{supg,h}(y_h,z_I)-A_{supg}(y_h,z_I)$. From the properties of the L^2-projection, (2.5), (3.8) and Lemmas 3.1-3.3, we arrive at

$$a_h(y_h,z_I)-a(y_h,z_I)$$
$$=\sum_{E\in\mathcal{T}_h}\Big(\varepsilon\big((\boldsymbol{\Pi}_{k-1}^0-I)\nabla y_h,\nabla z_I\big)_{0,E}+\big(\varepsilon+\tau_E\beta_E^2\big)S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)z_I\big)$$
$$+\tau_E\big(\boldsymbol{\beta}\cdot\boldsymbol{\Pi}_{k-1}^0\nabla y_h,\boldsymbol{\beta}\cdot(\boldsymbol{\Pi}_{k-1}^0-I)\nabla z_I\big)_{0,E}+\tau_E\big(\boldsymbol{\beta}\cdot(\boldsymbol{\Pi}_{k-1}^0-I)\nabla y_h,\boldsymbol{\beta}\cdot\nabla z_I\big)_{0,E}\Big)$$
$$\leq C\sum_{E\in\mathcal{T}_h}\Big(\varepsilon\big\|(\boldsymbol{\Pi}_{k-1}^0-I)\nabla y_h\big\|_{0,E}\|\nabla z_I\|_{0,E}+\tau_E\big(\boldsymbol{\beta}\cdot(\boldsymbol{\Pi}_{k-1}^0-I)\nabla y_h,\boldsymbol{\beta}\cdot\nabla z_I\big)_{0,E}$$
$$+\big(\varepsilon+\tau_E\beta_E^2\big)S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)^{\frac{1}{2}}\big\|\nabla(I-\Pi_k^\nabla)z_I\big\|_{0,E}$$
$$+\tau_E\big((I-\boldsymbol{\Pi}_{k-1}^0)\boldsymbol{\beta}\boldsymbol{\beta}^T\boldsymbol{\Pi}_{k-1}^0\nabla y_h,(\boldsymbol{\Pi}_{k-1}^0-I)\nabla z_I\big)_{0,E}\Big)$$
$$\leq C\bigg(\Big(\sum_{E\in\mathcal{T}_h}\varepsilon^2S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)\Big)^{\frac{1}{2}}\|\nabla z\|_{0,\Omega}$$
$$+\sum_{E\in\mathcal{T}_h}\tau_E\beta_E^2S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)^{\frac{1}{2}}h_E^{-1}\big(\|y_I-y\|_{0,E}+\|y-y_h\|_{0,E}\big)$$
$$+\sum_{E\in\mathcal{T}_h}\tau_E\big\|(I-\boldsymbol{\Pi}_{k-1}^0)\boldsymbol{\beta}\boldsymbol{\beta}^T\boldsymbol{\Pi}_{k-1}^0\nabla y_h\big\|_{0,E}h_E^{-1}\big(\|y_I-y\|_{0,E}+\|y-y_h\|_{0,E}\big)\bigg)$$
$$\leq C\bigg(\Big(\sum_{E\in\mathcal{T}_h}\varepsilon S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)\Big)^{\frac{1}{2}}\|z\|_{supg}$$
$$+\Big(\sum_{E\in\mathcal{T}_h}\big(h_E\beta_E+\delta^{-1}\beta_E^2\big)S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)\Big)^{\frac{1}{2}}\|y-y_h\|_{supg}$$

$$
\begin{aligned}
&+\bigg(\sum_{E\in\mathcal{T}_h}\beta_E^{-2}\big(\tau_E+\delta^{-1}\big)\big\|\big(I-\mathbf{\Pi}_{k-1}^0\big)\boldsymbol{\beta}\boldsymbol{\beta}^T\mathbf{\Pi}_{k-1}^0\nabla y_h\big\|_{0,E}^2\bigg)^{\frac{1}{2}}\|y-y_h\|_{supg}\bigg)\\
\le\; & C\|z\|_{supg}\bigg(\sum_{E\in\mathcal{T}_h}\Big(\big(\varepsilon+h_E\beta_E+\delta^{-1}\beta_E^2\big)S_a^E\big(\big(I-\Pi_k^\nabla\big)y_h,\big(I-\Pi_k^\nabla\big)y_h\big)\\
&\quad+\beta_E^{-2}\big(\tau_E+\delta^{-1}\big)\big\|\big(I-\mathbf{\Pi}_{k-1}^0\big)\boldsymbol{\beta}\boldsymbol{\beta}^T\mathbf{\Pi}_{k-1}^0\nabla y_h\big\|_{0,E}^2\Big)\bigg)^{\frac{1}{2}}.
\end{aligned}
$$

From the definitions of $b_h(y_h,z_I)$ and $b(y_h,z_I)$, we have

$$
\begin{aligned}
b_h(y_h,z_I)-b(y_h,z_I)\;&=\frac{1}{2}\bigg(\sum_{E\in\mathcal{T}_h}\Big(\big(\boldsymbol{\beta}\cdot\nabla\Pi_k^0y_h,\Pi_k^0z_I\big)_{0,E}-\big(\boldsymbol{\beta}\cdot\nabla y_h,z_I\big)_{0,E}\Big)\\
&\qquad+\sum_{s\in\mathcal{S}_h}\int_s[\![\boldsymbol{\beta}(I-\Pi_k^0)y_h\Pi_k^0z_I]\!]ds\bigg)\\
&\quad+\frac{1}{2}\bigg(\sum_{E\in\mathcal{T}_h}\Big(\big(y_h,\boldsymbol{\beta}\cdot\nabla z_I\big)_{0,E}-\big(\Pi_k^0y_h,\boldsymbol{\beta}\cdot\nabla\Pi_k^0z_I\big)_{0,E}\Big)\\
&\qquad-\sum_{s\in\mathcal{S}_h}\int_s\big[\!\big[\boldsymbol{\beta}\Pi_k^0y_h\big(I-\Pi_k^0\big)z_I\big]\!\big]ds\bigg)\\
&=:\frac{1}{2}b_1+\frac{1}{2}b_2.
\end{aligned}
$$

Next we estimate b_1 and b_2, respectively. From [14], we have the following inequality:

$$
\big\|\nabla\Pi_k^0\xi\big\|_{0,E}\le C\|\nabla\xi\|_{0,E} \tag{3.9}
$$

for all $\xi\in H^1(E)$. Therefore, using (3.9) and the triangular inequality, we further infer

$$
\begin{aligned}
\big\|\nabla\big(I-\Pi_k^0\big)y_h\big\|_{0,E}&=\big\|\nabla\big(I-\Pi_k^0\big)\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}\\
&\le\big\|\nabla\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}+\big\|\nabla\Pi_k^0\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}\\
&\le 2\big\|\nabla\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}.
\end{aligned}\tag{3.10}
$$

Then, by applying integration by parts, the properties of the L^2-projection, (3.8)-(3.10), Lemmas 3.1, 3.3 and a scaled Poincaré inequality we deduce

$$
\begin{aligned}
b_1=&\sum_{E\in\mathcal{T}_h}\Big(\big(\boldsymbol{\beta}\cdot\nabla\Pi_k^0y_h,\Pi_k^0z_I\big)_{0,E}-(\boldsymbol{\beta}\cdot\nabla y_h,z_I)_{0,E}+\big(\boldsymbol{\beta}\cdot\nabla\big(I-\Pi_k^0\big)y_h,\Pi_k^0z_I\big)_{0,E}\\
&\qquad+\big(\big(I-\Pi_k^0\big)y_h,\boldsymbol{\beta}\cdot\nabla\Pi_k^0z_I\big)_{0,E}\Big)\\
=&\sum_{E\in\mathcal{T}_h}\Big(\big(\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0y_h,\big(\Pi_k^0-I\big)z_I\big)_{0,E}+\big(\boldsymbol{\beta}\cdot\nabla\big(I-\Pi_k^0\big)y_h,\big(\Pi_k^0-I\big)z_I\big)_{0,E}\\
&\qquad+\big(\big(I-\Pi_k^0\big)y_h,\boldsymbol{\beta}\cdot\nabla\Pi_k^0z_I\big)_{0,E}\Big)\\
\le\; & C\sum_{E\in\mathcal{T}_h}\Big(\big\|\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0y_h\big\|_{0,E}\|z_I\|_{0,E}+h_E\beta_E\big\|\nabla\big(I-\Pi_k^0\big)y_h\big\|_{0,E}\|\nabla z_I\|_{0,E}\\
&\qquad+\beta_E\big\|\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}\big\|\nabla\Pi_k^0z_I\big\|_{0,E}\Big)\\
\le\; & C\sum_{E\in\mathcal{T}_h}\Big(\big\|\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0y_h\big\|_{0,E}\big(\|y_I-y\|_{0,E}+\|y-y_h\|_{0,E}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \beta_E \big\|\nabla\big(I-\Pi_k^{\nabla}\big)y_h\big\|_{0,E}\big(\|y_I-y\|_{0,E}+\|y-y_h\|_{0,E}\big)\Big)\\
\le{}& C\|z\|_{supg}\Bigg(\bigg(\sum_{E\in\mathcal{T}_h}\big(\tau_E+\delta^{-1}\big)\big\|\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0 y_h\big\|_{0,E}^2\bigg)^{\frac12}\\
&+\bigg(\sum_{E\in\mathcal{T}_h}\beta_E^2\big(\tau_E+\delta^{-1}\big)S_a^E\big(\big(I-\Pi_k^{\nabla}\big)y_h,\big(I-\Pi_k^{\nabla}\big)y_h\big)\bigg)^{\frac12}\Bigg)\\
\le{}& C\|z\|_{supg}\Bigg(\sum_{E\in\mathcal{T}_h}\Big(\big(\tau_E+\delta^{-1}\big)\big\|\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0 y_h\big\|_{0,E}^2\\
&+\beta_E^2\big(\tau_E+\delta^{-1}\big)S_a^E\big(\big(I-\Pi_k^{\nabla}\big)y_h,\big(I-\Pi_k^{\nabla}\big)y_h\big)\Big)\Bigg)^{\frac12}.
\end{aligned}
$$

Using the same computations yields

$$
\begin{aligned}
b_2\le C\|z\|_{supg}\Bigg(\sum_{E\in\mathcal{T}_h}&\Big(\beta_E^2\big(\tau_E+\delta^{-1}\big)S_a^E\big(\big(I-\Pi_k^{\nabla}\big)y_h,\big(I-\Pi_k^{\nabla}\big)y_h\big)\\
&+\big(\delta^{-1}+\tau_E\big)\big\|\big(I-\Pi_k^0\big)\boldsymbol{\beta}\cdot\nabla\Pi_k^0 y_h\big\|_{0,E}^2\Big)\Bigg)^{\frac12}.
\end{aligned}
$$

We may bound the term $b_h(y_h,z_I)-b(y_h,z_I)$ by combining with the above two estimates.

From the properties of the L^2-projection, Assumption 3.1, (2.5), (2.7) and the definition of norm $\|\cdot\|_{supg}$ we deduce

$$
\begin{aligned}
&c_h(y_h,z_I)-c(y_h,z_I)\\
={}&\sum_{E\in\mathcal{T}_h}\Big(\big(\delta\big(I-\Pi_k^0\big)y_h,\big(\Pi_k^0-I\big)z_I\big)_{0,E}+\tau_E\big(\delta\Pi_k^0 y_h,\boldsymbol{\beta}\cdot\big(\boldsymbol{\Pi}_{k-1}^0-I\big)\nabla z_I\big)_{0,E}\\
&+\tau_E\big(\delta\big(\Pi_k^0-I\big)y_h,\boldsymbol{\beta}\cdot\nabla z_I\big)_{0,E}+\delta S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)z_I\big)\Big)\\
\le{}& C\sum_{E\in\mathcal{T}_h}\Big(\delta\big\|\big(I-\Pi_k^0\big)y_h\big\|_{0,E}\|z_I\|_{0,E}+\tau_E\delta\big\|\big(I-\boldsymbol{\Pi}_{k-1}^0\big)\Pi_k^0 y_h\boldsymbol{\beta}\big\|_{0,E}\big\|\big(\boldsymbol{\Pi}_{k-1}^0-I\big)\nabla z_I\big\|_{0,E}\\
&+\tau_E\delta\big\|\big(I-\Pi_k^0\big)y_h\big\|_{0,E}\|\boldsymbol{\beta}\cdot\nabla z_I\|_{0,E}+\delta S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)y_h\big)^{\frac12}\big\|\big(I-\Pi_k^0\big)z_I\big\|_{0,E}\Big)\\
\le{}& C\Bigg(\bigg(\sum_{E\in\mathcal{T}_h}\big(\tau_E\delta^2+\delta\big)\big\|\big(I-\Pi_k^0\big)y_h\big\|_{0,E}^2\bigg)^{\frac12}\|y-y_h\|_{supg}\\
&+\bigg(\sum_{E\in\mathcal{T}_h}\beta_E^{-2}\big(\tau_E\delta^2+\delta\big)\big\|\big(I-\boldsymbol{\Pi}_{k-1}^0\big)\Pi_k^0 y_h\boldsymbol{\beta}\big\|_{0,E}^2\bigg)^{\frac12}\|y-y_h\|_{supg}\\
&+\bigg(\sum_{E\in\mathcal{T}_h}\big(\tau_E\delta^2+\delta\big)S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)y_h\big)\bigg)^{\frac12}\|y-y_h\|_{supg}\Bigg)\\
\le{}& C\|z\|_{supg}\Bigg(\sum_{E\in\mathcal{T}_h}\Big(\big(\tau_E\delta^2+\delta\big)S_b^E\big(\big(I-\Pi_k^0\big)y_h,\big(I-\Pi_k^0\big)y_h\big)\\
&+\beta_E^{-2}\big(\tau_E\delta^2+\delta\big)\big\|\big(I-\boldsymbol{\Pi}_{k-1}^0\big)\Pi_k^0 y_h\boldsymbol{\beta}\big\|_{0,E}^2\Big)\Bigg)^{\frac12}.
\end{aligned}
$$

For the last term, using the properties of the L^2-projection, (2.5)-(2.8) and (3.8), we derive

$$
\begin{aligned}
& d_h(y_h, z_I) - d(y_h, z_I) \\
&= \sum_{E\in\mathcal{T}_h} \Big(\tau_E \varepsilon \big(\nabla \cdot (I - \mathbf{\Pi}^0_{k-1})\nabla y_h, \boldsymbol{\beta} \cdot \nabla z_I\big)_{0,E} + \tau_E \varepsilon \big(\nabla \cdot \mathbf{\Pi}^0_{k-1}\nabla y_h, \boldsymbol{\beta} \cdot (I - \mathbf{\Pi}^0_{k-1})\nabla z_I\big)_{0,E} \Big) \\
&\le C \sum_{E\in\mathcal{T}_h} \Big(\tau_E \varepsilon \beta_E h_E^{-1} \big(C_k^E\big)^{-\frac{1}{2}} \big\| (I - \mathbf{\Pi}^0_{k-1})\nabla y_h \big\|_{0,E} \|\nabla z\|_{0,E} \\
&\qquad\qquad + \tau_E \varepsilon \big\| (I - \mathbf{\Pi}^0_{k-1}) (\nabla \cdot \mathbf{\Pi}^0_{k-1}\nabla y_h) \boldsymbol{\beta} \big\|_{0,E} \big\| (I - \mathbf{\Pi}^0_{k-1})\nabla z_I \big\|_{0,E} \Big) \\
&\le C \Bigg(\Big(\sum_{E\in\mathcal{T}_h} \tau_E^2 \varepsilon^2 \beta_E^2 h_E^{-2} \big(C_k^E\big)^{-1} \big\| \nabla (I - \Pi_k^\nabla) y_h \big\|^2_{0,E} \Big)^{\frac{1}{2}} \|\nabla z\|_{0,\Omega} \\
&\qquad + \Big(\sum_{E\in\mathcal{T}_h} \tau_E^2 \varepsilon^2 \big\| (I - \mathbf{\Pi}^0_{k-1}) (\nabla \cdot \mathbf{\Pi}^0_{k-1}\nabla y_h) \boldsymbol{\beta} \big\|^2_{0,E} \Big)^{\frac{1}{2}} \|\nabla z\|_{0,\Omega} \Bigg) \\
&\le C \|z\|_{supg} \Big(\sum_{E\in\mathcal{T}_h} \tau_E \beta_E^2 S_a^E \big((I - \Pi_k^\nabla) y_h, (I - \Pi_k^\nabla) y_h \big) + \tau_E^2 \varepsilon \big\| (I - \mathbf{\Pi}^0_{k-1}) (\nabla \cdot \mathbf{\Pi}^0_{k-1}\nabla y_h) \boldsymbol{\beta} \big\|^2_{0,E} \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then the result is obtained by combining the individual bounds above and using (2.7) again.

Remark 3.2. We can find that there are two extra terms on the right-hand side of (3.7) produced from the coercivity of A_{supg} in Theorem 3.1. It will be discussed here that these terms are in the convection dominated regime negligible compared with the error of the SUPG approximation. This point justifies the use of the quantity $(\sum_{E\in\mathcal{T}_h}(\eta^E + \Theta^E + \Xi^E + \Psi^E))^{1/2}$ as an upper error estimate.

The convergence rate of the SUPG approximation is $(r + 1/2), 0 < r \le k$, see (3.3). For the first term, one can apply Theorem 3.2 to get

$$
\sum_{E\in\mathcal{T}_h} \tau_E \varepsilon^2 \|\Delta(y - y_I)\|^2_{0,E} \le C \sum_{E\in\mathcal{T}_h} P_{e_E}^{-2} \beta_E h_E^{2r+1} \|y\|^2_{r+1,E}. \tag{3.11}
$$

For the second extra term on the right-hand side of (3.7), using (2.7) and Lemma 3.2, we also have

$$
\sum_{E\in\mathcal{T}_h} \tau_E \varepsilon^2 h_E^{-2} \|\nabla(y - y_I)\|^2_{0,E} \le C \sum_{E\in\mathcal{T}_h} P_{e_E}^{-2} \beta_E h_E^{2r+1} \|y\|^2_{r+1,E}. \tag{3.12}
$$

These bounds have a very small factor for $P_{e_E} \gg 1$ and possess the same asymptotic with respect to h like the left-hand side of (3.7). Then, the left-hand side of (3.11) and (3.12) become small compared with the left-hand side of (3.7), which shows the rationality of $(\sum_{E\in\mathcal{T}_h}(\eta^E + \Theta^E + \Xi^E + \Psi^E))^{1/2}$ as an upper bound error estimate.

3.3. Lower bound

We now prove a local lower bound of the a posteriori error estimate. To this end, we make use of element and edge bubble functions satisfying the bounds of Lemmas 3.5 and 3.6 respectively.

Theorem 3.4 (Local Lower Bound). *Let η^E, Θ^E and Ξ^E be given in Theorem 3.3, then there exist a positive constant C independent of h, τ_E, β_E and ε such that*

$$\eta^E \le C \sum_{E' \in \omega_E} \Big(\|y - y_h\|^2_{supg,E'} + \Theta^{E'} + \Xi^{E'} + \tau_{E'}\varepsilon^2 \|\Delta(y - y_I)\|^2_{0,E'} + \tau_{E'}\varepsilon^2 h_{E'}^{-2} \|\nabla(y - y_I)\|^2_{0,E'} \Big). \tag{3.13}$$

Here $\omega_E := \{E' \in \mathcal{T}_h : \mu_1(\partial E' \cap \partial E) \neq 0\}$ is the patch made up of the element E and its neighbours, and μ_1 is the one-dimensional measure.

Proof. Let $v = \Psi_E R_E, \lambda = 0$ in (3.6) and note that $\Psi_E|_{\partial E} = 0$, we infer

$$\begin{aligned} A_{supg}(z, \Psi_E R_E) = &(R_E, \Psi_E R_E)_{0,E} + \big(R_E, \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} \\ &+ \big(\theta_E, \Psi_E R_E + \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} \\ &+ B_1^E(y_h, -\Psi_E R_E) + B_2^E(y_h, -\Psi_E R_E). \end{aligned}$$

By the Lemma 3.5 we deduce

$$\begin{aligned} \|R_E\|^2_{0,E} &\le C_{bub}(R_E, \Psi_E R_E)_{0,E} \\ &= C_{bub}\Big(A^E_{supg}(z, \Psi_E R_E) - \big(R_E, \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} \\ &\qquad - \big(\theta_E, \Psi_E R_E + \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} + B_1^E(y_h, \Psi_E R_E) + B_2^E(y_h, \Psi_E R_E)\Big) \\ &=: C_{bub}\left(\sum_{i=1}^5 T_i\right). \end{aligned} \tag{3.14}$$

Using definition of A^E_{supg}, (2.7), (2.8), Lemmas 3.1 and 3.5, we have

$$\begin{aligned} T_1 = &\big(\varepsilon \nabla z, \nabla(\Psi_E R_E)\big)_{0,E} + \big(\boldsymbol{\beta} \cdot \nabla z, \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} + (\boldsymbol{\beta} \cdot \nabla z, \Psi_E R_E)_{0,E} \\ &+ \big(\delta z, \Psi_E R_E + \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} - \varepsilon\big(\Delta z, \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} \\ \le &\varepsilon \|\nabla z\|_{0,E} \|\nabla(\Psi_E R_E)\|_{0,E} + \tau_E \beta_E \|\boldsymbol{\beta} \cdot \nabla z\|_{0,E} \|\nabla(\Psi_E R_E)\|_{0,E} \\ &+ \|\boldsymbol{\beta} \cdot \nabla z\|_{0,E} \|\Psi_E R_E\|_{0,E} + \delta^{\frac12} \big\|\delta^{\frac12} z\big\|_{0,E} \|\Psi_E R_E\|_{0,E} \\ &+ \delta^{\frac12} \tau_E \beta_E \big\|\delta^{\frac12} z\big\|_{0,E} \|\nabla(\Psi_E R_E)\|_{0,E} + \varepsilon \tau_E \beta_E \|\Delta z\|_{0,E} \|\nabla(\Psi_E R_E)\|_{0,E} \\ \le &C\Big(\varepsilon^{\frac12} h_E^{-\frac12} \beta_E^{\frac12} \|\nabla z\|_{0,E} \|R_E\|_{0,E} + \|\boldsymbol{\beta} \cdot \nabla z\|_{0,E} \|R_E\|_{0,E} + \beta_E^{\frac12} h_E^{-\frac12} \big\|\delta^{\frac12} z\big\|_{0,E} \|R_E\|_{0,E} \\ &\qquad + \varepsilon \tau_E \beta_E h_E^{-1} \|R_E\|_{0,E} \big(\|\Delta(y - y_I)\|_{0,E} + \|\Delta(y_I - y_h)\|_{0,E}\big)\Big) \\ \le &C\Big(\varepsilon^{\frac12} h_E^{-\frac12} \beta_E^{\frac12} \|\nabla z\|_{0,E} \|R_E\|_{0,E} + \|\boldsymbol{\beta} \cdot \nabla z\|_{0,E} \|R_E\|_{0,E} + \beta_E^{\frac12} h_E^{-\frac12} \|\delta^{\frac12} z\|_{0,E} \|R_E\|_{0,E} \\ &\qquad + \varepsilon \|R_E\|_{0,E} \|\Delta(y - y_I)\|_{0,E} + \varepsilon \|R_E\|_{0,E} h_E^{-1} \big(\|\nabla(y_I - y)\|_{0,E} + \|\nabla z\|_{0,E}\big)\Big) \\ \le &C \tau_E^{-\frac12} \|R_E\|_{0,E} \Big(\|z\|_{supg,E} + \varepsilon \tau_E^{\frac12} \|\Delta(y - y_I)\|_{0,E} + \varepsilon h_E^{-1} \tau_E^{\frac12} \|\nabla(y_I - y)\|_{0,E}\Big). \end{aligned} \tag{3.15}$$

Applying (2.7) and Lemma 3.5, we have the following estimates for T_2 and T_3:

$$\begin{aligned} T_2 = &-\big(R_E, \tau_E \boldsymbol{\beta} \cdot \nabla(\Psi_E R_E)\big)_{0,E} \le \tau_E \beta_E \|R_E\|_{0,E} \|\nabla(\Psi_E R_E)\|_{0,E} \\ &\le C_{bub} \tau_E \beta_E h_E^{-1} \|R_E\|^2_{0,E} \le \frac{1}{2C_{bub}} \|R_E\|^2_{0,E}, \end{aligned} \tag{3.16}$$

$$T_3 \le \|\theta_E\|_{0,E} \|\Psi_E R_E\|_{0,E} + \tau_E \beta_E \|\theta_E\|_{0,E} \|\nabla(\Psi_E R_E)\|_{0,E} \le C \|\theta_E\|_{0,E} \|R_E\|_{0,E}.$$

From the definitions of B_1^E, (2.5), (2.7) and Lemma 3.5, we arrive at

$$\begin{aligned}
&B_1^E(y_h, \Psi_E R_E) \\
&\le \varepsilon \big\|\nabla\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}\|\nabla(\Psi_E R_E)\|_{0,E} + \beta_E\big\|\nabla\big(I-\Pi_k^\nabla\big)y_h\big\|_{0,E}\|\Psi_E R_E\|_{0,E} \\
&\quad + \delta\big\|\big(I-\Pi_k^0\big)y_h\big\|_{0,E}\|\Psi_E R_E\|_{0,E} \\
&\le C\|R_E\|_{0,E}\Big((\varepsilon h_E^{-1}+\beta_E)S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)^{\frac12} + \delta S_b^E\big((I-\Pi_k^0)y_h,(I-\Pi_k^0)y_h\big)^{\frac12}\Big) \\
&\le C\|R_E\|_{0,E}\Big(\beta_E^2 S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big) + \delta^2 S_b^E\big((I-\Pi_k^0)y_h,(I-\Pi_k^0)y_h\big)\Big)^{\frac12} \\
&\le C\|R_E\|_{0,E}\Big(\delta\big(\beta_E^2\delta^{-1}S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big) + \delta S_b^E\big((I-\Pi_k^0)y_h,(I-\Pi_k^0)y_h\big)\big)\Big)^{\frac12} \\
&\le C\|R_E\|_{0,E}\Big(\beta_E h_E^{-1}\big(\beta_E^2\delta^{-1}S_a^E\big((I-\Pi_k^\nabla)y_h,(I-\Pi_k^\nabla)y_h\big)+\delta S_b^E\big((I-\Pi_k^0)y_h,(I-\Pi_k^0)y_h\big)\big)\Big)^{\frac12} \\
&\le C\|R_E\|_{0,E}\tau_E^{-\frac12}\Xi_E^{\frac12}.
\end{aligned}\tag{3.17}$$

Using similar computations of (3.17) and inverse inequality (2.8), we can also deduce

$$B_2^E(y_h, \Psi_E R_E) \le C\|R_E\|_{0,E}\tau_E^{-\frac12}\Xi_E^{\frac12}. \tag{3.18}$$

Combining the above estimates, we get the bound for the element residual as follows:

$$\begin{aligned}
\frac12\|R_E\|_{0,E}^2 \le C\|R_E\|_{0,E}\tau_E^{-\frac12}\Big(&\|z\|_{supg,E} + \tau_E^{\frac12}\|\theta_E\|_{0,E} + \Xi_E^{\frac12} + \varepsilon\tau_E^{\frac12}\|\Delta(y-y_I)\|_{0,E} \\
&+ \varepsilon h_E^{-1}\tau_E^{\frac12}\|\nabla(y_I-y)\|_{0,E}\Big).
\end{aligned}$$

Further, we have

$$\begin{aligned}
\tau_E\|R_E\|_{0,E}^2 \le C\big(&\|z\|_{supg,E}^2 + \tau_E\|\theta_E\|_{0,E}^2 + \Xi_E + \varepsilon^2\tau_E\|\Delta(y-y_I)\|_{0,E}^2 \\
&+ \varepsilon^2 h_E^{-2}\tau_E\|\nabla(y_I-y)\|_{0,E}^2\big).
\end{aligned}$$

$\omega_s := E^+ \cup E^-$ with E^+ and E^- the elements meeting at the edge s. We extend J_s into ω_s through a constant prolongation in the normal direction of the edge s for the edge residual. Let $v = \Psi_s J_s$ in (3.6) we deduce

$$\begin{aligned}
A_{supg}(z, \Psi_s J_s) = \sum_{E'\in\omega_s}\Big(&\big(R_{E'}, \Psi_s J_s + \tau_{E'}\boldsymbol{\beta}\cdot\nabla(\Psi_s J_s)\big)_{0,E'} + \big(\theta_{E'}, \Psi_s J_s + \tau_{E'}\boldsymbol{\beta}\cdot\nabla(\Psi_s J_s)\big)_{0,E'} \\
&+ B_1^{E'}(y_h, -\Psi_s J_s) + B_2^{E'}(y_h, -\Psi_s J_s)\Big) + (J_s, -\Psi_s J_s)_{0,s}.
\end{aligned}$$

By Lemma 3.6, we have

$$\begin{aligned}
\|J_s\|_{0,s}^2 &\le C_{bub}(J_s, \Psi_s J_s)_{0,s} \\
&= C_{bub}\sum_{E'\in\omega_s}\Big(\big(R_{E'}, \Psi_s J_s + \tau_{E'}\boldsymbol{\beta}\cdot\nabla(\Psi_s J_s)\big)_{0,E'} + \big(\theta_{E'}, \Psi_s J_s + \tau_{E'}\boldsymbol{\beta}\cdot\nabla(\Psi_s J_s)\big)_{0,E'} \\
&\qquad + B_1^{E'}(y_h, -\Psi_s J_s) + B_2^{E'}(y_h, -\Psi_s J_s)\Big) - C_{bub}A_{supg}^{\omega_s}(z, \Psi_s J_s).
\end{aligned}$$

Arguing as (3.15)-(3.18) and using Lemma 3.6, we find that

$$\begin{aligned}
\|J_s\|_{0,s}^2 \le C\sum_{E'\in\omega_s}\Big(&\beta_{E'}^{\frac12}\|z\|_{supg,E'}\|J_s\|_{0,s} + h_{E'}^{\frac12}\|R_{E'}\|_{0,E'}\|J_s\|_{0,s} + h_{E'}^{\frac12}\|\theta_{E'}\|_{0,E'}\|J_s\|_{0,s} \\
&+ \beta_{E'}^{\frac12}\Xi_{E'}^{\frac12}\|J_s\|_{0,s} + h_{E'}^{\frac12}\varepsilon\|\Delta(y-y_I)\|_{0,E'}\|J_s\|_{0,s} \\
&+ \varepsilon h_{E'}^{-\frac12}\|\nabla(y_I-y)\|_{0,E'}\|J_s\|_{0,s}\Big).
\end{aligned}$$

Using the estimate of element residual, we arrive at

$$\begin{aligned}\|J_s\|_{0,s}^2 \le& C\sum_{E'\in\omega_s}\Big(\beta_{E'}\|z\|_{supg,E'}^2+\beta_{E'}\tau_{E'}\|\theta_{E'}\|_{0,E'}^2+\beta_{E'}\Xi_{E'}\\ &+\beta_{E'}\tau_{E'}\varepsilon^2\|\Delta(y-y_I)\|_{0,E'}^2+\beta_{E'}\tau_{E'}\varepsilon^2h_{E'}^{-2}\|\nabla(y_I-y)\|_{0,E'}^2\Big)\\ \le& C\beta_{\omega_s}\sum_{E'\in\omega_s}\Big(\|z\|_{supg,E'}^2+\tau_{E'}\|\theta_{E'}\|_{0,E'}^2+\Xi_{E'}\\ &+\tau_{E'}\varepsilon^2\|\Delta(y-y_I)\|_{0,E'}^2+\tau_{E'}\varepsilon^2h_{E'}^{-2}\|\nabla(y_I-y)\|_{0,E'}^2\Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\beta_{\omega_E}^{-1}\sum_{s\subset\partial E}\|J_s\|_{0,s}^2\le& C\sum_{E'\in\omega_E}\Big(\|z\|_{supg,E'}^2+\tau_{E'}\|\theta_{E'}\|_{0,E'}^2+\Xi_{E'}\\ &+\tau_{E'}\varepsilon^2\|\Delta(y-y_I)\|_{0,E'}^2+\tau_{E'}\varepsilon^2h_{E'}^{-2}\|\nabla(y_I-y)\|_{0,E'}^2\Big).\end{aligned}$$

By putting the above bounds together the theorem is complete. □

Remark 3.3. Reasoning in the same way in Remark 3.2, one finds that the additional items are negligible compared with $\|y-y_h\|_{supg,E'}$ in (3.13).

For each element $E\in\mathcal{T}_h$, we define the local error estimator $\eta_h(y_h,E)$ by

$$\eta_h(y_h,E):=\left(\eta^E+\Theta^E+\Xi^E+\Psi^E\right)^{\frac12}.$$

Then on a subset $\omega\subseteq\Omega$, we define the error estimator $\eta_h(y_h,\omega)$ by

$$\eta_h(y_h,\omega):=\Big(\sum_{E\in\mathcal{T}_h,E\subset\omega}\eta_h^2(y_h,E)\Big)^{\frac12}.$$

Thus, the error indicator on Ω with respect to $\mathcal{T}_h$ can be expressed as $\eta_h(y_h,\Omega)$.

4. Numerical Results

In this section, we firstly introduce an adaptive VEM algorithm based on a posteriori error estimate to solve the problem (2.4). Then we carry out some numerical tests to verify our theoretical analysis.

4.1. Algorithm

Here we consider the adaptive VEM with $k=1$. In the first step, we solve the problem (2.4) on a given initial mesh $\mathcal{T}_{h_0}$ with mesh size h_0. Then we calculate the indicator $\eta_h(y_h,\Omega)$. In the third step, we using Dörfler's marking strategy to construct and mark the set of elements that need to be refined. Finally, we refine them. For the refinement strategy, we adopt the idea in [17], which connects the barycentre with the midpoint of each edge in every marked element, as shown in Fig. 4.1.

The concrete adaptive VEM algorithm is given below.

In addition, we shall test the performance of the indicator by computing the following effective index:

$$\text{effectivity}:=\frac{\eta_h(y_h,\Omega)}{\|y-y_h\|_{supg}}.$$

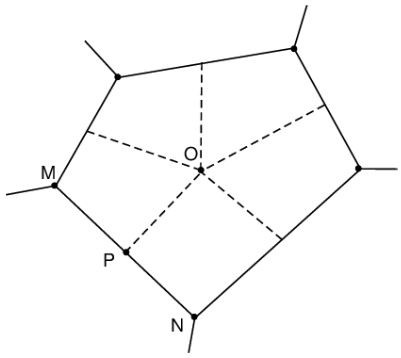


Fig. 4.1. In each refinement, the barycentre of the every marked element, i.e. O, and the midpoint of each edge are introduced as new nodes, for example P.

Algorithm 4.1: Adaptive VEM Algorithm.

Require: The tolerance error κ of the indicator and a coarse mesh $\mathcal{T}_{h_0}$ with mesh size h_0.

Ensure :

1 Set $error = 2\kappa$ and $t = 0$.

2 **while** $error > \kappa$ **do**

3 Start with the mesh $\mathcal{T}_{h_t}$ with mesh size h_t and corresponding virtual element spaces $V_{h_t,1}$.

4 Solve the problem (2.4) on the actual mesh $\mathcal{T}_{h_t}$ for y_h.

5 Calculate the local error indicator $\eta_{h_t}(y_h, E)$ for each element $E \in \mathcal{T}_{h_t}$.

6 Set $error = \eta_{h_t}(y_h, \Omega)$ and $t = t + 1$.

7 Evaluate stopping criterion, that is, **if** $error \leq \kappa$, **then** stop adaptive iteration **else** go to next step.

8 Construct a minimal subset $\mathcal{M}_{h_{t-1}} \subset \mathcal{T}_{h_{t-1}}$ such that

$$\sum_{E \in \mathcal{M}_{h_t}} \eta^2_{h_{t-1}}(y_h, E) \geq \mu \eta^2_{h_{t-1}}(y_h, \Omega), \quad 0 < \mu < 1.$$

9 Mark all the elements in $\mathcal{M}_{h_{t-1}}$.

10 Refine $\mathcal{M}_{h_{t-1}}$ to get a new mesh $\mathcal{T}_{h_t}$ with mesh size h_t and go to Step 3.

11 **end**

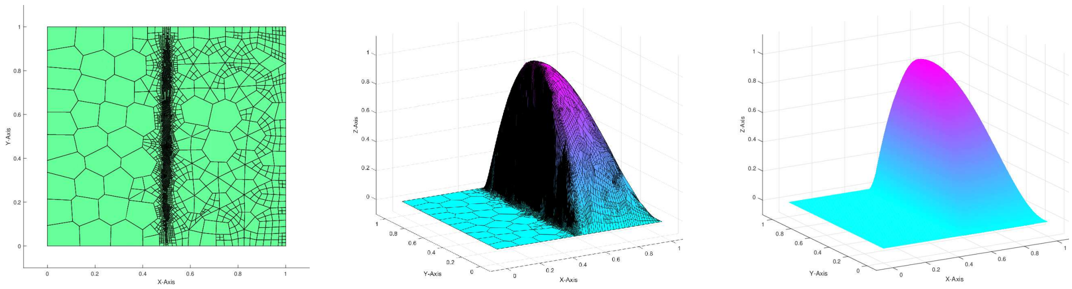
4.2. Numerical experiments

Example 4.1 (Solution with Interior Layer). Problem (2.1) is considered in $\Omega = (0,1)^2$ with $\boldsymbol{\beta} = (1,0)^T, \delta = 1$, and with f being chosen such that

$$y(x_1, x_2) = \frac{25}{3} x_1(1 - x_1) x_2(1 - x_2) \left(1 - \tanh \frac{\gamma - x_1}{\zeta}\right)$$

is the solution of (2.1).

The parameters γ and ζ control the location and thickness of the interior layer. Here we choose $\gamma = 0.5, \zeta = 0.01$ and the diffusion coefficient $\varepsilon = 10^{-6}$ and 10^{-10}, respectively. The



(a) (b) (c)

Fig. 4.2. Example 4.1: (a) The adapted Voronoi mesh after 25 iterations with $\varepsilon = 10^{-10}$ and $\mu = 0.4$, (b) The profile of the numerical solution y_h on the finally refined Voronoi mesh with $\varepsilon = 10^{-10}$, (c) Exact solution.

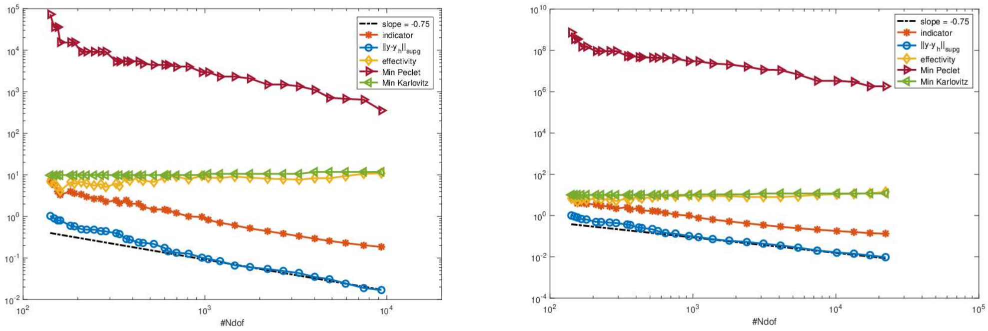


Fig. 4.3. Example 4.1: The convergence history of the error, error indicator and effectivity on adaptively refined Voronoi meshes with $\varepsilon = 10^{-6}, \mu = 0.3$ (left) and $\varepsilon = 10^{-10}, \mu = 0.4$ (right) generated by Algorithm 4.1.

solution has a sharp decrease along $x_1 = 0.5$. It is well known that the numerical solution shows oscillations on uniform refinements. The adaptive VEM algorithm is applied to improve numerical performance in the computing. We set marking parameter $\mu = 0.3$ and $\mu = 0.4$, respectively. Fig. 4.2 shows the adapted Voronoi mesh [32] after 25 iterations with $\varepsilon = 10^{-10}$, where we find that mesh refinement is almost concentrated in the interior layer. We also display the exact solution and the numerical solution of based on the adapted Voronoi mesh with $\varepsilon = 10^{-10}$, which shows a very good agreement with the exact solution.

For each adaptive mesh we report the values of the minimum mesh Péclet and Karlovitz numbers in Fig. 4.3 with $\varepsilon = 10^{-6}$ and 10^{-10}, respectively. We can observe that they are always much bigger than 1. From (2.9), we can know the whole adaptive process is completed in convection dominated regime. Results of the SUPG error $\|y - y_h\|_{supg}$, error indicator $\eta_h(y_h, \Omega)$ and the effectivity of the indicator are also shown in Fig. 4.3 with $\varepsilon = 10^{-6}$ and 10^{-10}. Therein, we observe that the convergence rates of both errors with respect to Ndof are optimal at $\mathcal{O}(\text{Ndof}^{-3/4})$ when $k = 1$. Furthermore, we see from the Fig. 4.3 that the efficiency index tends to be a constant through the mesh sequence.

Example 4.2 (Solution with Circular Interior Layer). This example is defined by $\Omega = (0,1)^2, \varepsilon = 10^{-6}, \boldsymbol{\beta} = (2,3)^T, \delta = 1$ and f such that

$$y(x_1, x_2) = 16x_1(1 - x_1)x_2(1 - x_2)\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{2\left(0.25^2 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2\right)}{\sqrt{\varepsilon}}\right]\right).$$

In this example, the exact solution y has interior layer along the circle $(x_1-0.5)^2-(x_2-0.5)^2 = 0.25^2$. We test the reliability and effectivity of the indicator $\eta_h(y_h,\Omega)$ on the square mesh. In Fig. 4.4, the profiles of the numerically computed solution for $\varepsilon = 10^{-6}$ and exact solution are shown, respectively. We can find that the oscillations in the solution are eliminated. The finally adapted mesh with $\varepsilon = 10^{-6}, \mu = 0.6$ is given in Fig. 4.5. We can see that the mesh is concentrated on the internal sharpening layer. The reliability of adaptive VEM algorithm is well verified.

We present the convergence of the error $\|y-y_h\|_{supg}$, error indicator $\eta_h(y_h,\Omega)$ and effectivity of the indicator on the square mesh with $\varepsilon = 10^{-6}, \mu = 0.6$ in Fig. 4.5. We can observe that the convergence order of error $\|y-y_h\|_{supg}$ is approximately parallel to the line with slope $-3/4$. With the continuous adaptive refinement of the grid, the effectivity of the indicator gradually tends to a constant. The data results are in agreement with the theoretical prediction.

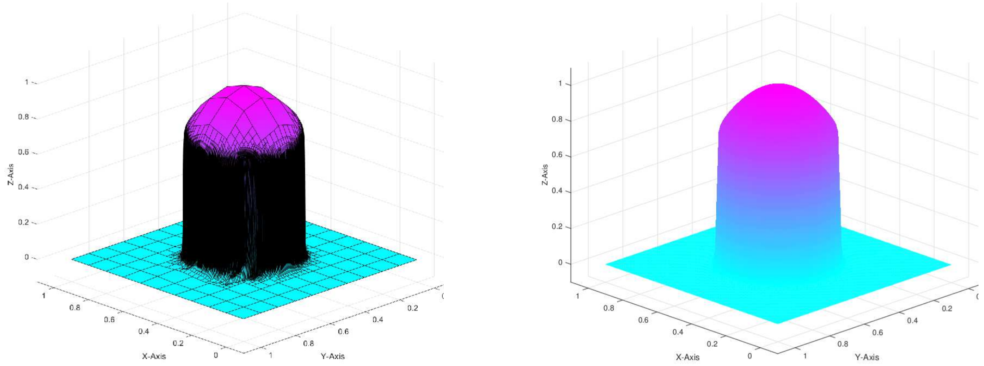


Fig. 4.4. Example 4.2: The profiles of the numerically computed solution for $\varepsilon = 10^{-6}$ (left) and exact solution (right).

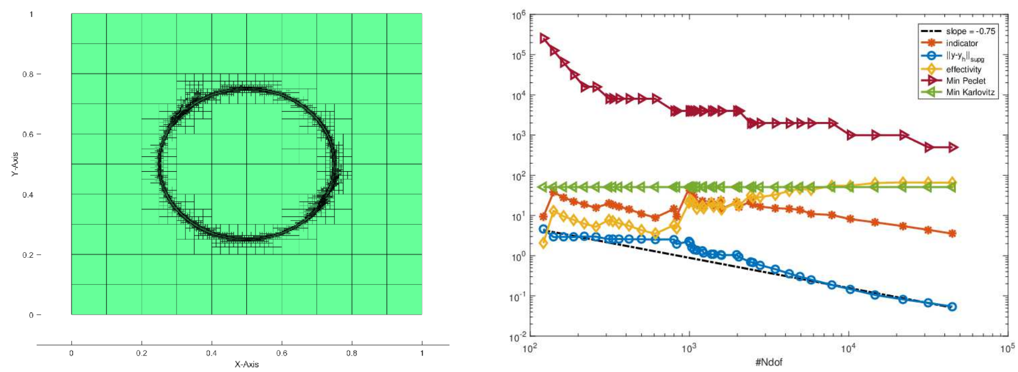


Fig. 4.5. Example 4.2: The finally adaptive square mesh after 40 iterations with $\varepsilon = 10^{-6}, \mu = 0.6$ (left) and the convergence history of the error, error indicator and effectivity on adaptively refined square meshes (right).

Example 4.3 (Solution with Boundary Layer). In this example, we take $\Omega = (0,1)^2$, $\varepsilon = 10^{-4}, \boldsymbol{\beta} = (2,2)^T, \delta = 1$. The right-hand side f is chosen so that

$$y(x_1,x_2) = \left(x_1 - \frac{e^{(x_1-1)/\varepsilon} - e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}}\right)\left(x_2 - \frac{e^{(x_2-1)/\varepsilon} - e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}}\right).$$

In this example, the exact y has boundary layers at $x_1 = 1$ and $x_2 = 1$. We employ the posteriori error indicator $\eta_h(y_h, \Omega)$ to construct adaptive mesh for y. The finally adaptive meshes with $\mu = 0.3$ are shown in Fig. 4.6. We can see that the meshes are strongly refined along the boundary $x_1 = 1$ and $x_2 = 1$, which indicates that the posteriori error estimator indicator $\eta_h(y_h, \Omega)$ can effectively capture the boundary layers. The profiles of numerical solution y_h on the distorted square mesh and exact solution are presented in Fig. 4.7, which represent and process the boundary layers well by comparison.

Fig. 4.8 shows the convergence behaviours of the error $\|y - y_h\|_{supg}$, error indicator $\eta_h(y_h, \Omega)$ and effectivity of the indicator on the distorted square and Voronoi meshes. We can see that in the case of convection dominated regime convergence order of error $\|y - y_h\|_{supg}$ is approximately parallel to the line with slope $-3/4$ which is the optimal convergence rate and with the continuous refinement of the mesh, effectivity asymptotically tends a constant.

Example 4.4 (Solution with Interior and Boundary Layers). The last test is a classic problem from [24]. In [11, 13], the SUPG stabilization for the conforming and nonconforming virtual element method are used to investigate this problem. Whether conforming VEM or nonconforming VEM, the solution obtained by SUPG method presents oscillations. Here adaptive VEM is used to study this problem. The geometry of the problem is depicted in Fig. 4.9.

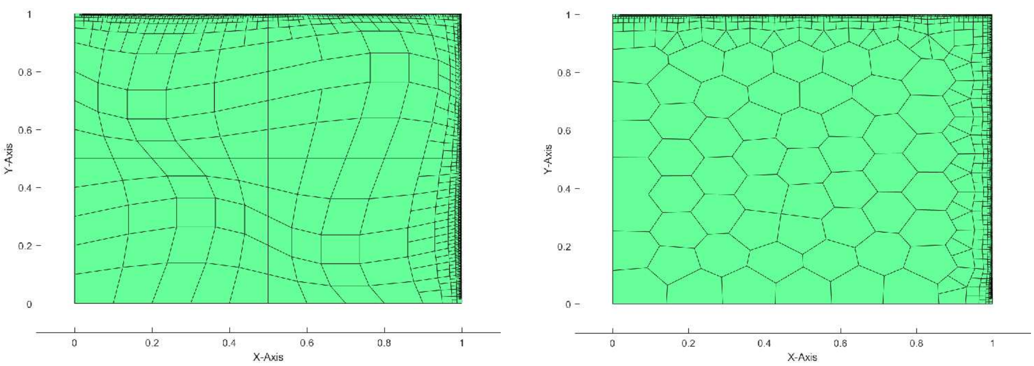


Fig. 4.6. Example 4.3: The finally adaptively distorted square mesh after 41 iterations (left) and Voronoi mesh after 42 iterations (right) with $\varepsilon = 10^{-4}$ and $\mu = 0.3$.

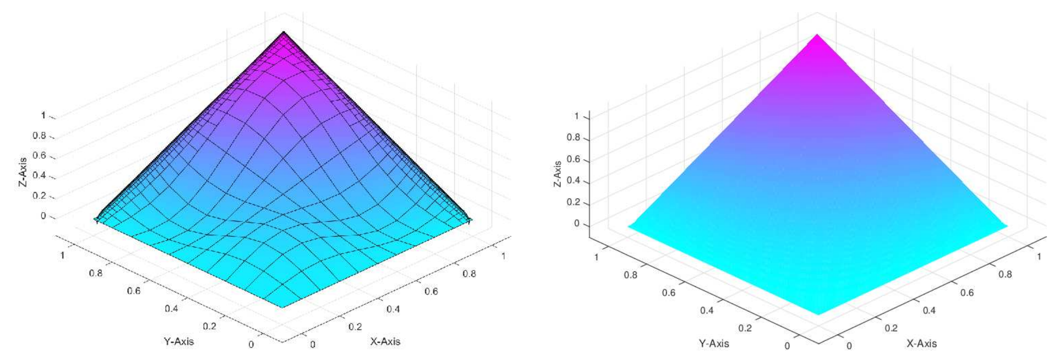


Fig. 4.7. Example 4.3: The profiles of the numerical solution y_h on adaptively distorted square mesh (left) with $\varepsilon = 10^{-4}$ and exact solution (right).

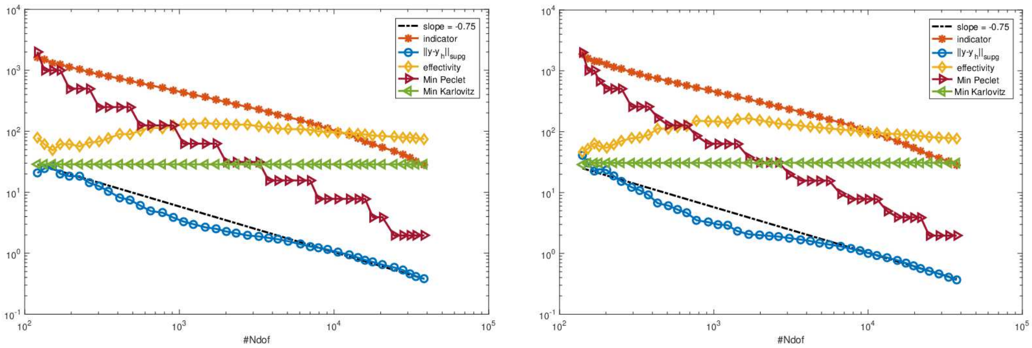


Fig. 4.8. Example 4.3: The convergence history of the error, error indicator and effectivity on adaptively distorted square mesh (left) and Voronoi mesh (right) with $\varepsilon = 10^{-4}$ generated by Algorithm 4.1.

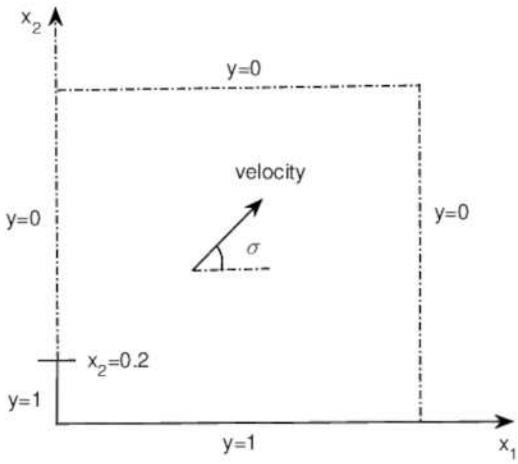


Fig. 4.9. Example 4.4: Domain and boundary conditions.

The velocity forms an angle σ with the x-axis, and propagates the non-homogeneous boundary condition $y = 1$ inside Ω, thus generating an internal discontinuity, which is numerically approximated by an internal layer, a sharp transition between the constant solution states $y = 0$ and 1. The homogeneous boundary condition at the top of the computational domain produces a boundary layer. Data for the problem is: $\varepsilon = 10^{-5}$, $\mu = 0.5$, $\boldsymbol{\beta} = (\cos(\sigma), \sin(\sigma))$ with $\sigma = \arctan(1)$.

We chose the distorted hexagonal mesh and the square mesh as the initial mesh. Fig. 4.10 shows the initial solutions and the solutions after adaptive refinement with $\varepsilon = 10^{-5}$. We can find that the initial results Figs. 4.10(a) and 4.10(d) are polluted with spurious oscillations in a vicinity of layers. From Figs. 4.10(c) and 4.10(f), it is clear that mesh refinement is all concentrated in the boundary and interior layers, which is consistent with our expected refinement behavior. With the refinement of the mesh, Figs. 4.10(b) and 4.10(e) show that the huge oscillations of the unstable solution are gradually reduced.

Acknowledgments. This work was partially supported by the National Natural Science Foundation of China (Grant Nos. 11971276, 11301311) and by the Natural Science Foundation of Shandong Province (Grant No. ZR2016JL004).

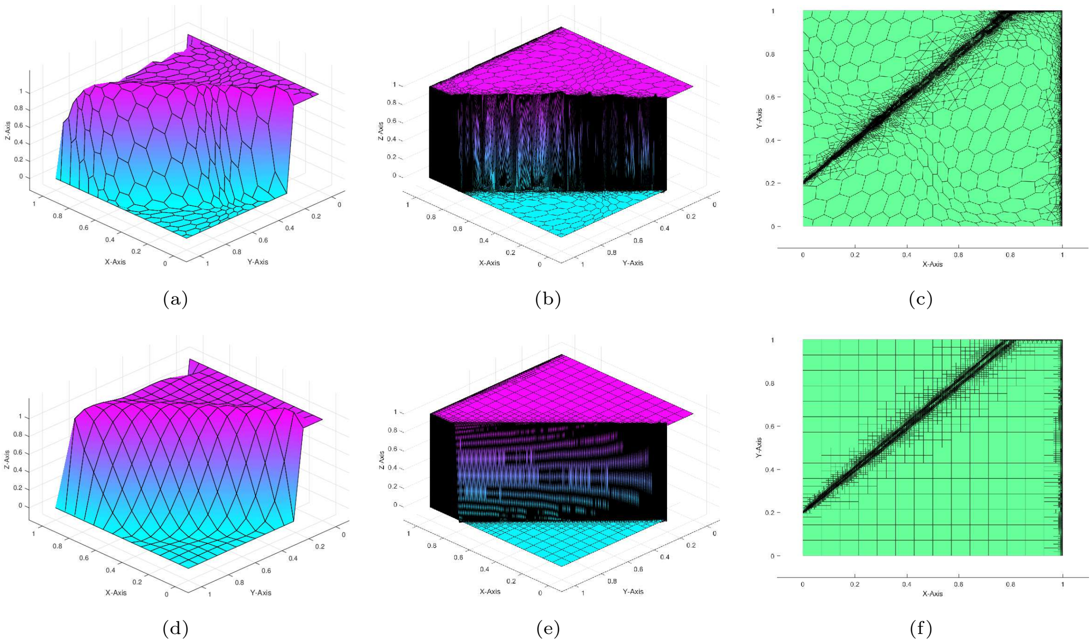


Fig. 4.10. Example 4.4: (a) The initial solution on distorted hexagonal mesh with $\varepsilon = 10^{-5}$, (b) The profile of final solution on distorted hexagonal mesh with $\varepsilon = 10^{-5}$, (c) The finally adaptively distorted hexagonal mesh with $\varepsilon = 10^{-5}$, (d) The initial solution on square mesh with $\varepsilon = 10^{-5}$, (e) The profile of final solution on square mesh with $\varepsilon = 10^{-5}$, (f) The finally adaptive square mesh with $\varepsilon = 10^{-5}$.

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