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UNIFORM ERROR BOUNDS OF AN ENERGY-PRESERVING EXPONENTIAL WAVE INTEGRATOR FOURIER PSEUDO-SPECTRAL METHOD FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH WAVE OPERATOR AND WEAK NONLINEARITY*

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Abstract

Recently, the numerical methods for long-time dynamics of PDEs with weak nonlinearity have received more and more attention. For the nonlinear Schrödinger equation (NLS) with wave operator (NLSW) and weak nonlinearity controlled by a small value $\varepsilon \in (0,1]$, an exponential wave integrator Fourier pseudo-spectral (EWIFP) discretization has been developed (Guo *et al.*, 2021) and proved to be uniformly accurate about ε up to the time at $\mathcal{O}(1/\varepsilon^2)$. However, the EWIFP method is not time symmetric and can not preserve the discrete energy. As we know, the time symmetry and energy-preservation are the important structural features of the true solution and we hope that this structure can be inherited along the numerical solution. In this work, we propose a time symmetric and energy-preserving exponential wave integrator Fourier pseudo-spectral (SEPEWIFP) method for the NLSW with periodic boundary conditions. Through rigorous error analysis, we establish uniform error bounds of the numerical solution at $\mathcal{O}(h^{m_0} + \varepsilon^{2-\beta}\tau^2)$ up to the time at $\mathcal{O}(1/\varepsilon^{\beta})$ for $\beta \in [0,2]$, where h and τ are the mesh size and time step, respectively, and m_0 depends on the regularity conditions. The tools for error analysis mainly include cut-off technique and the standard energy method. We also extend the results on error bounds, energy-preservation and time symmetry to the oscillatory NLSW with wavelength at $\mathcal{O}(\varepsilon^2)$ in time which is equivalent to the NLSW with weak nonlinearity. Numerical experiments confirm that the theoretical results in this paper are correct. Our method is novel because that to the best of our knowledge there has not been any energy-preserving exponential wave integrator method for the NLSW.

Mathematics subject classification: 35Q55, 65M12, 65M15, 65M70, 81-08.
Key words: Nonlinear Schrödinger equation with wave operator and weak nonlinearity, Fourier pseudo-spectral method, Exponential wave integrator, Energy-preserving method, Error estimates, Oscillatory problem.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation with wave operator in d $(d = 1,2,3)$ dimensions on a torus $\mathbb{T}^d$:

$$\begin{cases} i\partial_t u(\mathbf{x},t) - \alpha\partial_{tt}u(\mathbf{x},t) + \Delta u(\mathbf{x},t) - \varepsilon^2|u(\mathbf{x},t)|^2u(\mathbf{x},t) = 0, & \mathbf{x}\in\mathbb{T}^d, \quad t>0, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \partial_t u(\mathbf{x},0) = u_1(\mathbf{x}), & \mathbf{x}\in\mathbb{T}^d, \end{cases} \tag{1.1}$$

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where $\mathbf{x} \in \mathbb{T}^d$ is the spatial coordinate, t is time, Δ is Laplacian, $\alpha > 0, u := u(\mathbf{x}, t)$ is a complex-valued scalar field, $\varepsilon \in (0, 1]$ is a dimensionless parameter characterizing the nonlinear strength, the functions $u_0(\mathbf{x})$ and $u_1(\mathbf{x})$ are complex-valued and independent of ε [2,6,11,15,34,35,37,39]. The NLSW (1.1) has different physical applications, including the nonrelativistic limit of the Klein-Gordon (KG) equation [34,35,37], the Langmuir wave envelope approximation in plasma [11,15], and the modulated planar pulse approximation of the sine-Gordon equation for light bullets [6,39]. The NLSW (1.1) is time symmetric and preserves the mass and the energy as

$$\begin{aligned} M(t) &:= \int_{\mathbb{T}^d} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} - 2\alpha \int_{\mathbb{T}^d} \mathrm{Im}\left[\overline{u(\mathbf{x},t)}\partial_t u(\mathbf{x},t)\right] \mathrm{d}\mathbf{x} := M(0), \quad t \geq 0, \\ E(t) &:= \int_{\mathbb{T}^d} \left[\alpha|\partial_t u(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2 + \frac{1}{2}\varepsilon^2 |u(\mathbf{x},t)|^4\right] \mathrm{d}\mathbf{x} := E(0), \quad t \geq 0, \end{aligned} \tag{1.2}$$

where $\overline{c}$ and $\mathrm{Im}(c)$ denote the conjugate and imaginary part of c, respectively.

By introducing $w(\mathbf{x}, t) = \varepsilon u(\mathbf{x}, t)$, we can reformulate the NLSW (1.1) with weak nonlinearity as the following NLSW with small initial data at $\mathcal{O}(\varepsilon)$:

$$\begin{cases} i\partial_t w(\mathbf{x},t) - \alpha\partial_{tt} w(\mathbf{x},t) + \Delta w(\mathbf{x},t) - |w(\mathbf{x},t)|^2 w(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{T}^d, \quad t > 0, \\ w(\mathbf{x},0) = \varepsilon u_0(\mathbf{x}), \quad \partial_t w(\mathbf{x},0) = \varepsilon u_1(\mathbf{x}), & \mathbf{x} \in \mathbb{T}^d. \end{cases} \tag{1.3}$$

Again, the above NLSW (1.3) is time symmetric and preserves the mass and the energy as

$$\begin{aligned} \widetilde{M}(t) &:= \int_{\mathbb{T}^d} |w(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} - 2\alpha \int_{\mathbb{T}^d} \mathrm{Im}\left[\overline{w(\mathbf{x},t)}\partial_t w(\mathbf{x},t)\right] \mathrm{d}\mathbf{x} := \widetilde{M}(0), \quad t \geq 0, \\ \widetilde{E}(t) &:= \int_{\mathbb{T}^d} \left[\alpha|\partial_t w(\mathbf{x},t)|^2 + |\nabla w(\mathbf{x},t)|^2 + \frac{1}{2}|w(\mathbf{x},t)|^4\right] \mathrm{d}\mathbf{x} := \widetilde{E}(0), \quad t \geq 0. \end{aligned} \tag{1.4}$$

Due to that the Eqs. (1.1) and (1.3) are equivalent, in the following, we only present numerical methods and related analysis for the NLSW (1.1) with weak nonlinearity. For the NLSW (1.3) with small initial data, the formulation of the new method and the analysis process are completely similar.

When $\alpha = 0$, the NLSW (1.1) reduces to the nonlinear Schrödinger equation. There are various numerical methods for NLS in the literature, including the time-splitting pseudospectral method [3, 10, 12, 16, 26, 33], the finite difference method [1, 3, 4, 13], etc. Meanwhile, for fixed $\varepsilon = 1$, there are some numerical methods for the NLSW, such as conservative finite difference methods [14, 15, 23, 28, 38] and exponential wave integrator method [2, 5]. Conservative finite difference methods are very popular due to that they can preserve the discrete energy and mass. In the recent work [2, 5], two finite difference methods (CNFD and SIFD) and an exponential wave integrator sine pseudospectral (EWISP) method have been analyzed for NLSW with $\alpha \to 0, \varepsilon = 1$ and proved to have different uniform error estimates with respect to $\alpha \in (0, 1]$ for well-prepared initial data and for ill-prepared initial data, respectively. For the research on numerical methods of NLSW in other parameter regimes, we see [40].

Recently, the numerical methods for long-time dynamics of PDEs with weak nonlinearity have received more and more attention. The long-time dynamics of the Klein-Gordon (KG) equations and Dirac equations with weak nonlinearity or small potential are thoroughly studied in the literature [7, 8, 18, 20–22, 30–32]. For the weak nonlinear NLSW with periodic boundary condition, an exponential wave integrator Fourier pseudo-spectral method has been proposed in [24] and proved to be uniformly accurate about ε up to the time at $\mathcal{O}(1/\varepsilon^2)$. Numerical