

RECONSTRUCTION-BASED A POSTERIORI ERROR ESTIMATES FOR THE L1 METHOD FOR TIME FRACTIONAL PARABOLIC PROBLEMS*

Jiliang Cao

*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China;
Hunan Key Laboratory for Computation and Simulation in Science and Engineering & Key
Laboratory of Intelligent Computing and Information Processing of Ministry of Education,
Xiangtan University, Xiangtan 411105, China*

Aiguo Xiao

*Hunan Key Laboratory for Computation and Simulation in Science and Engineering & Key
Laboratory of Intelligent Computing and Information Processing of Ministry of Education,
Xiangtan University, Xiangtan 411105, China*

Email: xag@xtu.edu.cn

Wansheng Wang¹⁾

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

Email: w.s.wang@163.com

Abstract

In this paper, we study a posteriori error estimates of the L1 scheme for time discretizations of time fractional parabolic differential equations, whose solutions have generally the initial singularity. To derive optimal order a posteriori error estimates, the quadratic reconstruction for the L1 method and the necessary fractional integral reconstruction for the first-step integration are introduced. By using these continuous, piecewise time reconstructions, the upper and lower error bounds depending only on the discretization parameters and the data of the problems are derived. Various numerical experiments for the one-dimensional linear fractional parabolic equations with smooth or nonsmooth exact solution are used to verify and complement our theoretical results, with the convergence of α order for the nonsmooth case on a uniform mesh. To recover the optimal convergence order $2 - \alpha$ on a nonuniform mesh, we further develop a time adaptive algorithm by means of barrier function recently introduced. The numerical implementations are performed on nonsmooth case again and verify that the true error and a posteriori error can achieve the optimal convergence order in adaptive mesh.

Mathematics subject classification: 65M15, 65M50, 65M06, 65M12, 35R11, 26A33.

Key words: Time fractional parabolic differential equations, A posteriori error estimates, L1 method, Fractional integral reconstruction, Quadratic reconstruction.

1. Introduction

Adaptive methods have become very popular and powerful tools for certain classes of PDEs. A posteriori error analysis can provide information about the error introduced by discretization and is at the base of adaptive computation. Time adaptive algorithms are naturally related to error control and variable time step-sizes. In this paper we derive a posteriori error estimates

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¹⁾ Corresponding author

for time discretization by the L1 method for abstract time fractional parabolic differential equations (TFPDEs) and construct an adaptive algorithm based on this rigorous a posteriori error estimates. To do this, we first introduce functional space and corresponding norms.

1.1. Functional space and norms

Let H be a Hilbert space with inner product $(\cdot, \cdot)$. We identify H with its dual. Let $A: D(A) \rightarrow H$ be a positive definite, self-adjoint, linear operator on H with domain $D(A)$ being dense in H . Let $V := D(A^{1/2})$ and denote the norms in H and V by $\|\cdot\|$ and $\|\cdot\|_1$, $\|v\|_1 = \|A^{1/2}v\| = (Av, v)^{1/2}$, respectively. Let V^* be the dual of V , and denote by $\|\cdot\|_{-1}$ the dual norm on V^* , $\|v\|_{-1} = \|A^{-1/2}v\| = (v, A^{-1}v)^{1/2}$. We still denote by $(\cdot, \cdot)$ the duality pairing between V^* and V . In a natural way the Lebesgue spaces $L^p(J; X)$ with a time interval $J = [t_*, t^*]$ and Banach space X (here, $X = H, V$ or V^*), $1 \leq p < \infty$, consist of all those functions $u(t)$ that take values in X for almost every $t \in J$ such that the L^p norm of $\|u(t)\|_X$, i.e.

$$\|u\|_{L^p(J; X)} = \left(\int_J \|u(t)\|_X^p dt \right)^{\frac{1}{p}}$$

is finite. For $p = \infty$, $L^\infty(J; X)$ is the space of (classes of) measurable functions from J into X which are essentially bounded, the space is Banach for the norm

$$\|u\|_{L^\infty(J; X)} := \operatorname{ess\,sup}_{t \in J} \|u(t)\|_X.$$

Note that for continuous function $u(t)$, we have $\|u\|_{L^\infty(J; X)} = \max_{t \in J} \|u(t)\|_X$. For simplicity, we will write $L^p(0, t; X)$ for $L^p((0, t); X)$.

Let ∂_t^α denote the Caputo fractional derivative of order α ($0 < \alpha < 1$) with respect to t defined by

$$\partial_t^\alpha u(t) := \int_0^t \omega_{1-\alpha}(t-s)u'(s)ds, \quad \omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

where $\Gamma(z)$ is the Gamma function,

$$\Gamma(z) := \int_0^\infty s^{z-1}e^{-s}ds, \quad \Re(z) > 0.$$

We define the Riemann-Liouville fractional integral operator of order β ($\beta \geq 0$) as

$$I_t^\beta u(t) = \int_0^t \omega_\beta(t-s)u(s)ds, \quad t > 0$$

with $I_t^0 u(t) = u(t)$ [44]. Then we recall the relationship between the Riemann-Liouville fractional derivative and Caputo fractional derivative

$$\partial_t^\alpha u(t) = \frac{d}{dt} \{ I_t^{1-\alpha} [u(t) - u(0)] \} = \frac{d}{dt} \left\{ \int_0^t \omega_{1-\alpha}(t-s)[u(s) - u(0)]ds \right\},$$

and the relationship of the Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative [54]

$$\partial_t^{-\alpha} u(t) = I_t^\alpha u(t) = \int_0^t \omega_\alpha(t-s)u(s)ds, \quad t > 0. \quad (1.1)$$