Journal of Computational Mathematics
Vol.43, No.2, 2025, 369–393.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2210-m2022-0057

TIME MULTIPOINT NONLOCAL PROBLEM FOR A STOCHASTIC SCHRÖDINGER EQUATION*

Roger Pettersson[1)]
Department of Mathematics, Linnaeus University, Växjö, Sweden
Email: roger.pettersson@lnu.se
Ali Sirma
Department of Industrial Engineering, Halic University, Istanbul, Turkey
Email: alisirma@halic.edu.tr
Tarkan Aydin
Department of Computer Engineering, Bahcesehir University, Istanbul, Turkey
Email: tarkan.aydin@bau.edu.tr

Abstract

A time multipoint nonlocal problem for a Schrödinger equation driven by a cylindrical Q-Wiener process is presented. The initial value depends on a finite number of future values. Existence and uniqueness of a solution formulated as a mild solution is obtained. A single-step implicit Euler-Maruyama difference scheme, a Rothe-Maryuama scheme, is suggested as a numerical solution. Convergence rate for the solution of the difference scheme is established. The theoretical statements for the solution of this difference scheme is supported by a numerical example.

Mathematics subject classification: 60H20, 60H35, 35J10, 65H10.
Key words: Time nonlocal problem, Mild solution, Cylindrical Wiener process, Time discretization, Abstract time-dependent stochastic Schrödinger equation, Euler-Maruyama method.

1. Introduction

Typically, phenomena evolving in time in various fields such as natural sciences, engineering, and finance are described in terms of differential equations. In models for which uncertainty is needed to build into the model, inherent randomness is a natural additional ingredient.

The most well-known differential equations subjected to randomness are stochastic ordinary differential equations of which among the most profound example model stock prices, see e.g. [28]. Partial differential equations with uncertainty can also be handled as stochastic partial differential equations, see e.g. [26] where applications to environmental pollution models and bond market models appear. In the above examples the initial value is typically independent of the time-evolving random noise. For backward stochastic differential equations, suitable for stochastic control and option pricing, the final value is a random variable adapted to the filtration at the final time point where the solution is nevertheless non-anticipating, see e.g. [23, Section 3]. In [22, Chapter 3.3] a stochastic two point boundary problem is considered. It is a finite-dimensional linear Stratonovich stochastic differential equation where the initial value depends linearly on the final value and is therefore anticipating, existence and uniqueness

* Received March 10, 2022 / Revised version received June 15, 2022 / Accepted October 10, 2022 / Published online January 29, 2024 /
1) Corresponding author

of a solution is shown. For a shooting method as a numerical solution to a stochastic two-point boundary value problem, see [6].

Existence and uniqueness of solution to evolution equations with nonlocal bondary conditions, formulated as abstract nonlocal Cauchy problems, are typically shown by fixed point arguments, see e.g. [2,3,11,16]; in [3] an application is given by a diffusion of a small amount of gas in a transparent tube which allows measurements at different time points and not just at time zero. In [8], a deterministic Schrödinger equation is studied where the initial value is a linear combination of future values, i.e. there are non-local boundary conditions in time. Existence, uniqueness, as well as stability of a numerical approximation is obtained. Also in [17,18], infinite dimensional time-nonlocal problems for deterministic Schrödinger type equations is considered. In [33] various applications of deterministic time-nonlocal dynamical systems are reviewed, also including Schrödinger equations. Time-nonlocal dynamical systems allow couplings of the initial conditions with the system in a nonlocal manner, rather than at a single point [33]. Note that time-nonlocal problems are generalisations of for instance periodic conditions, see e.g. [32]. The most applications refereed to are in quantum mechanics where for instance it can be seen as a way to mitigate the influence of initial conditions, or include the ability to impose initial and final boundary conditions on the evolution of a quantum system. For nonlocal in time problems applied to radionuclides propagation in Stokes fluid and problems of predicting the state of a medium see e.g. references in [9].

A stochastically dispersed Schrödinger equation with a linear diffusive term is known as the Belavkin equation; for a rigorous treatment of such equations see e.g. [25]. For analysis of stochastic Schrödinger equations with cubic nonlinear drift subjected to a multiplicative finite-dimensional Wiener process and standard initial value condition, see for instance [1, 10, 15, 30]; in [30] and its references, applications appear in optical fiber communication; in [15] stability of finite element approximation in space combined with various time discretization schemes such as explicit and implicit Euler and Crank-Nicolson schemes is obtained where the noise is of Stratonovich type and the equation is formulated in a variational weak form; for more references of numerics of stochastic nonlinear Schrödinger equations with given initial value see for instance the references in [15]. In [5], an infinite-dimensional Q-Wiener process is allowed where for the case of linear stochastic Schrödinger equation, temporal discretization convergence of an exponential integrator scheme is obtained of order one for additive noise and $1/2$ for multiplicative noise. In [12–14] strong and weak convergence rates of several numerical schemes for stochastic nonlinear Schrödinger equations with non-monotone coefficients and multiplicative noise with given initial value are derived. In [31], numerics for deterministic nonlocal-in-time Schrödinger equations is considered.

In this paper, a stochastic Schrödinger equation with a time-dependent Gaussian excitation and time non-local initial condition is considered, which, to our knowledge, is novel in combining temporal discretization of a stochastic Schrödinger equation subjected to a cylindrical Q-Wiener process with time multi-point initial condition. Since the initial value is a linear combination of future values, the solution is not adapted to the given filtration. Here the drift is linear and the dispersion is non-anticipating. That makes it possible to formulate a solution within the framework of Itô-integrals in infinite dimensions, here in a mild form, which can be compared to the finite-dimensional linear two-point boundary stochastic differential equation [22] where a mild form is not needed.

The involved operator in [7] is self-adjoint positive definite while in [8] and in this paper the operator is only assumed to be self-adjoint. In [7], writing the equation in a mild form, the