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ORACLE INEQUALITIES FOR CORRUPTED COMPRESSED SENSING*

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Abstract

In this paper, we establish the oracle inequalities of highly corrupted linear observations $\mathbf{b} = \mathbf{A}\mathbf{x}_0 + \mathbf{f}_0 + \mathbf{e} \in \mathbb{R}^m$. Here the vector $\mathbf{x}_0 \in \mathbb{R}^n$ with $n \gg m$ is a (approximately) sparse signal and $\mathbf{f}_0 \in \mathbb{R}^m$ is a sparse error vector with nonzero entries that can be possible infinitely large, $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}_m)$ represents the Gaussian random noise vector. We extend the oracle inequality $\|\hat{\mathbf{x}} - \mathbf{x}_0\|_2^2 \lesssim \sum_i \min\{|x_0(i)|^2, \sigma^2\}$ for Dantzig selector and Lasso models in [E.J. Candès and T. Tao, Ann. Statist., 35 (2007), 2313–2351] and [T.T. Cai, L. Wang, and G. Xu, IEEE Trans. Inf. Theory, 56 (2010), 3516–3522] to $\|\hat{\mathbf{x}}-\mathbf{x}_0\|_2^2+\|\hat{\mathbf{f}}-\mathbf{f}_0\|_2^2 \lesssim \sum_i \min\{|x_0(i)|^2, \sigma^2\} + \sum_j \min\{|\lambda f_0(j)|^2, \sigma^2\}$ for the extended Dantzig selector and Lasso models. Here $(\hat{\mathbf{x}}, \hat{\mathbf{f}})$ is the solution of the extended model, and $\lambda > 0$ is the balance parameter between $\|\mathbf{x}\|_1$ and $\|\mathbf{f}\|_1$, i.e. $\|\mathbf{x}\|_1 + \lambda\|\mathbf{f}\|_1$.

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1. Introduction

1.1. Corrupted compressed sensing problem

Over the past twenty years, the idea of compressed sensing has received extensive attention and has been employed in several potential technologies [8, 10]. It offers an excellent strategy for reconstructing a (approximately) sparse signal from a few observations. In particular, an s-sparse signal $\mathbf{x}_0 \in \mathbb{R}^n$ is evaluated by

$$\mathbf{b} = \mathbf{A}\mathbf{x}_0 + \mathbf{e}, \tag{1.1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m \ll n$ is the sensing matrix, $\mathbf{b} \in \mathbb{R}^m$ denotes the observation vector and $\mathbf{e} \in \mathbb{R}^m$ is the possible noise vector.

The following optimization problem:

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{x}\|_0 \\ &\text{s.t.} \quad \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{C}(\eta) \end{aligned}$$

provides a good estimator for the reconstruction of $\mathbf{x}_0$. Here $\|\mathbf{x}\|_0 = |\{i : x_i \neq 0\}|$ expresses the sparsity of $\mathbf{x}$, $\mathcal{C}(\eta)$ is a bounded set with the parameter $\eta > 0$ determined by the error

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structure, for example, $\mathcal{C}(\eta) = \{\mathbf{z} : \|\mathbf{z}\|_2 \leq \eta\}$ or $\mathcal{C}(\eta) = \{\mathbf{z} : \|\mathbf{A}^\top \mathbf{z}\|_\infty \leq \eta\}$ [11]. Here and following, we use the notation $\mathbf{A}^\top \in \mathbb{R}^{n\times m}$ denotes the transposition of the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$. There exist some convex methods to solve this optimization problem. The method of basis pursuit [13, 14] transformed the ℓ_0-minimization $\|\mathbf{x}\|_0$ to its relative convex ℓ_1-minimization $\|\mathbf{x}\|_1$ ($\|\mathbf{x}\|_1 = \sum_i |x_i|$), solved the non-deterministic polynomial (NP) hard problem. Candès and Tao [9] proved that the original signal $\mathbf{x}_0$ can be exactly recovered by solving that ℓ_1-minimization problem. Based on this, a number of methods for different noise types have been proposed, such as Lasso [41], quadratically constrained basis pursuit [18], Dantzig selector [11], and RLAD [44,47]. Extensive studies appear under different frameworks, such as the null space property [17, 22, 39], the restricted isometry property (RIP) [5, 15, 16, 49], and the coherence [4, 19, 28, 29, 42], solving this problem.

When certain unknown items of the observation vector are badly distorted, we can get a novel method inspired by the above classic compressed sensing issue. In mathematics, we have

$$\mathbf{b} = \mathbf{A}\mathbf{x}_0 + \mathbf{f}_0 + \mathbf{e}. \tag{1.2}$$

Here $\mathbf{f}_0 \in \mathbb{R}^m$ is a corrupted error, which is unidentified and cannot be disregarded. Corrupted compressed sensing is the issue of reconstructing the sparse signal $\mathbf{x}_0$ and sparse error $\mathbf{f}_0$ from the observations (1.2). Laska *et al.* [25] first considered recovering the signal and the corruption from corrupted measurements and designed an algorithm dubbed Justice Pursuit. They extended the classical RIP to the generalized restricted isometry property (GRIP) as follows.

Definition 1.1. *For any matrix $\mathbf{\Phi} = [\mathbf{A}, \mathbf{I}] \in \mathbb{R}^{m\times(n+m)}$, the (s,t)-GRIP-constant $\delta_{s,t}$ is defined as the infimum of δ such that*

$$(1-\delta)\left(\|\mathbf{x}\|_2^2 + \|\mathbf{f}\|_2^2\right) \leq \left\|\mathbf{\Phi}\begin{bmatrix}\mathbf{x}\\ \mathbf{f}\end{bmatrix}\right\|_2^2 \leq (1+\delta)\left(\|\mathbf{x}\|_2^2 + \|\mathbf{f}\|_2^2\right)$$

holds for any $\mathbf{x} \in \mathbb{R}^n$ with $|\mathrm{supp}(\mathbf{x})| \leq s$ and $\mathbf{f} \in \mathbb{R}^m$ with $|\mathrm{supp}(\mathbf{f})| \leq t$.

As a nontrivial extension of compressed sensing, the corrupted compressed sensing problem has been used in various practical fields, such as super-resolution and inpainting [33], signal recovery from the impulsive observations [36], signal separation [21].

In recent years, many breakthroughs have been obtained in the research of the corrupted compressed sensing problem. In the absence of the noise $\mathbf{e}$, Wright and Ma [45] proposed to recover the signal $\mathbf{x}_0$ and the corruption $\mathbf{f}_0$ from the observations $\mathbf{b}$ in (1.2) by solving the following problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{f}\in\mathbb{R}^m} \|\mathbf{x}\|_1 + \|\mathbf{f}\|_1 \quad \text{s.t.} \quad \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{f}.$$

Considering the general situation with random noise $\mathbf{e}$, being tiny, Lin and Li [31] proposed to recover the sparse signal from the corrupted observations (1.2) with coherent tight frames via separation analysis Dantzig selector (SADS)

$$\min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{f}\in\mathbb{R}^m} \|\mathbf{D}^\top\mathbf{x}\|_1 + \|\mathbf{\Omega}^\top\mathbf{f}\|_1 \quad \text{s.t.} \quad \left\|\mathbf{W}^\top[\mathbf{A}, \mathbf{I}]^\top(\mathbf{A}\mathbf{x} + \mathbf{f} - \mathbf{b})\right\|_\infty \leq \eta,$$