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BINARY LEAST SQUARES: AN ALGORITHM FOR BINARY SPARSE SIGNAL RECOVERY*

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Abstract

A fundamental problem in some applications including group testing and communications is to acquire the support of a K-sparse signal $\boldsymbol{x}$, whose nonzero elements are 1, from an underdetermined noisy linear model. This paper first designs an algorithm called binary least squares (BLS) to reconstruct $\boldsymbol{x}$ and analyzes its complexity. Then, we establish two sufficient conditions for the exact reconstruction of $\boldsymbol{x}$'s support with K iterations of BLS based on the mutual coherence and restricted isometry property of the measurement matrix, respectively. Finally, extensive numerical tests are performed to compare the efficiency and effectiveness of BLS with those of batch orthogonal matching pursuit (Batch-OMP) which to our best knowledge is the fastest implementation of OMP, orthogonal least squares (OLS), compressive sampling matching pursuit (CoSaMP), hard thresholding pursuit (HTP), Newton-step-based iterative hard thresholding (NSIHT), Newton-step-based hard thresholding pursuit (NSHTP), binary matching pursuit (BMP) and ℓ_1-regularized least squares. Test results show that: (1) BLS can be 10-200 times more efficient than Batch-OMP, OLS, CoSaMP, HTP, NSIHT and NSHTP with higher probability of support reconstruction, and the improvement can be 20%-80%; (2) BLS has more than 25% improvement on the support reconstruction probability than the explicit BMP algorithm with a little higher computational complexity; (3) BLS is around 100 times faster than ℓ_1-regularized least squares with lower support reconstruction probability for small K and higher support reconstruction probability for large K. Numerical tests on the generalized space shift keying (GSSK) detection indicate that although BLS is a little slower than BMP, it is more efficient than the other seven tested sparse recovery algorithms, and although it is less effective than ℓ_1-regularized least squares, it is more effective than the other seven algorithms.

Mathematics subject classification: 94A12, 65F22, 65J22.
Key words: Binary sparse signal, Precise support reconstruction, Binary least squares.

1. Introduction

The reconstruction of an unknown K-sparse $\boldsymbol{x}$ (i.e. $\boldsymbol{x}$ has no more than K nonzero entries) from the following underdetermined noisy linear measurements lies at the heart of compressive sensing [8,12]:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}, \tag{1.1}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ is a given measurement vector, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is a given measurement matrix satisfying $m \ll n$, and $\boldsymbol{w} \in \mathbb{R}^m$ is a ℓ_2-bounded noise vector, i.e., $\boldsymbol{w}$ satisfies $\|\boldsymbol{w}\|_2 \le \epsilon$ for certain small constant ϵ. There are other kinds of noises, for further details, see, e.g. [5,37].

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Although we assume $\boldsymbol{w}$ is a ℓ_2-bounded noise, the results in this work cannot be hard to be extended to other types of noise by following some techniques in e.g. [5, 36].

The problem of stably acquiring the K-sparse $\boldsymbol{x}$ from (1.1) arises in a massive number of applications, such as magnetic resonance imaging [23] and radar imaging [27]. While in some other applications, such as group testing [17], generalized space shift keying modulation detection [19] and active user detection [10], in addition to be K-sparse, $\boldsymbol{x}$ also fulfills

$$x_i = 1 \quad \text{for} \quad i \in \Omega, \tag{1.2}$$

where $\Omega = \{i | x_i \neq 0\}$ is the support of $\boldsymbol{x}$. Note that K-sparse $\boldsymbol{x}$ satisfying (1.2) is called binary K-sparse signal.

Although there is a large number of algorithms for reconstructing $\boldsymbol{x}$ from (1.1), such as greedy algorithms [11, 25, 26, 32], convex optimization algorithms [8, 12] and thresholding algorithms [2, 4, 30], and there are some studies on the reconstruction of binary sparse signals from (structured) biased measurement matrices [14, 20], there are few algorithms specifically designed for acquiring binary K-sparse $\boldsymbol{x}$ from (1.1) for any measurement matrix $\boldsymbol{A}$, and binary matching pursuit (BMP) [34] is the most recent one.

As explained in [34], designing an efficient and effective algorithm for reconstructing $\boldsymbol{x}$ satisfying (1.2) is of vital importance. Note that the reconstruction of such kind of sparse signals is challenging for orthogonal matching pursuit (OMP) [32] and it has special interest for the comparative study as they represent a particularly challenging case for OMP-type of reconstruction strategies [11]. Furthermore, it is emphasized in [3] that the recovery of sparse signals with equal magnitude nonzero entries is most demanding, and it was conjectured in [38] that the most difficult sparse inverse problem may involve nonzero coefficients with equal magnitudes. [18, Theorem 1] supports the observation, that reconstructing sparse vectors with equal magnitude nonzero coefficients correspond to the most difficult case for many recovery algorithms, by stating that, as long as the satisfaction of mutual coherence conditions for exact recovery is concerned, "flat" vectors (i.e. vectors whose nonzero entries are a constant) correspond to the worst possible case for OMP and OLS.

This work focuses on designing an efficient and effective binary sparse signal reconstruction algorithm and studying its performance. More exactly, we develop an iterative reconstruction algorithm called binary least squares (BLS). In each iteration, as orthogonal least squares (OLS) [9], BLS selects an index such that the residual vector is shortest. But different from OLS which mathematically solves a least squares problem to find the index, it uses (1.2) to find the index and does not solve any least squares problem, hence it is much more efficient than OLS. The new algorithm is a variant of OLS, and since it is designed for acquiring binary sparse signals, we call it BLS.

Although BMP is an iterative algorithm which also uses (1.2) to iteratively find the support of $\boldsymbol{x}$ in each iteration, its selection criterion is different from that of BLS. More exactly, in each iteration, BMP selects an index such that the absolute value of the inner product of the corresponding column of $\boldsymbol{A}$ and the current residual vector is maximized, which is different from that of BLS. Since a natural method to recover $\boldsymbol{x}$ is to minimize $\|\boldsymbol{x}\|_0$ subject to $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2 \leq \epsilon$, where $\|\boldsymbol{x}\|_0$ denotes the number of nonzero entries of $\boldsymbol{x}$, the selection strategy of BLS is closer to this method than that of BMP, hence BLS is expected to have better recovery performance than BMP. In fact, BLS is much more effective than BMP with a little higher complexity. Further details on this will be presented in Section 4.