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CENTRAL LIMIT THEOREM FOR TEMPORAL AVERAGE OF BACKWARD EULER-MARUYAMA METHOD*

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Abstract

This work focuses on the temporal average of the backward Euler-Maruyama (BEM) method, which is used to approximate the ergodic limit of stochastic ordinary differential equations (SODEs). We give the central limit theorem (CLT) of the temporal average of the BEM method, which characterizes its asymptotics in distribution. When the deviation order is smaller than the optimal strong order, we directly derive the CLT of the temporal average through that of original equations and the uniform strong order of the BEM method. For the case that the deviation order equals to the optimal strong order, the CLT is established via the Poisson equation associated with the generator of original equations. Numerical experiments are performed to illustrate the theoretical results. The main contribution of this work is to generalize the existing CLT of the temporal average of numerical methods to that for SODEs with super-linearly growing drift coefficients.

Mathematics subject classification: 60H35, 60F05, 60H10.
Key words: Central limit theorem, Temporal average, Ergodicity, Backward Euler-Maruyama method.

1. Introduction

Ergodic theory is a powerful tool to investigate the long-time dynamics and statistical properties of stochastic systems, which is widely applied in physics, biology and chemistry (see e.g. [8, 13, 30, 33]). A crucial problem in ergodic theory is to determine the ergodic measure and ergodic limit. Since explicit expressions of them are generally unavailable, one usually resorts to numerical methods to obtain their approximations. There have been lots of numerical methods which inherit the ergodicity or approximate the ergodic limit of original systems (see [1, 12, 14, 24, 27, 31] and references therein). In the aforementioned work, main efforts are made to analyze the error between the numerical invariant measure and the original one, and that between numerical temporal average and the ergodic limit.

Besides the convergence of the numerical temporal average in the moment sense, the asymptotics of its distribution is also an essential property. In recent several work, the central limit theorem (CLT) of the temporal average of some numerical methods is given, which characterizes the fluctuation of the numerical temporal average around ergodic limits of original systems in the sense of distribution. In [26], the CLT of the temporal average of the Euler-Maruyama (EM) method with decreasing step-size for ergodic stochastic ordinary differential equations (SODEs) is given. In addition, [23] proves the CLT and moderate deviation of the EM method

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with a fixed step-size for SODEs. For a class of ergodic stochastic partial differential equations (SPDEs), [6] shows that the temporal average of a full discretization with fixed temporal and spatial step-sizes satisfies the CLT.

In the existing work, the CLT of numerical temporal average is established provided that coefficients of original equations are Lipschitz continuous. Compared with the Lipschitz case, stochastic systems with non-Lipschitz coefficients have more extensive applications in reality (see e.g. [3, 7, 9, 11] and references therein). For example, consider the overdamped Langevin equation

$$\mathrm{d}q(t) = -\nabla V\big(q(t)\big)\mathrm{d}t + \sqrt{2\beta^{-1}}\,\mathrm{d}W(t), \tag{1.1}$$

where $\{W(t), t \geq 0\}$ is a D-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, and β^{-1} is the Boltzmann constant times the absolute temperature. The potential V is smooth and satisfies $\lim_{|q|\to+\infty} V(q) = +\infty$ (for example, one can take $D = 1$ and $V(q) = q^4/4 + q^2/2$, $q \in \mathbb{R}$). The Langevin equation describes the noise-induced transport in stochastic ratchets and dissipative particle dynamics. When the inertia of the particle is negligible compared with the damping force due to the friction, the trajectory of the Langevin equation is approximately described by (1.1) (see e.g. [17, 28, 29]). It is known that (1.1) admits a unique invariant measure (thus is ergodic) $\pi(\mathrm{d}q) = Z^{-1}e^{-\beta V(q)}\mathrm{d}q$ with $Z = \int_{\mathbb{R}} e^{-\beta V(q)}\mathrm{d}q$. Since the drift coefficient of (1.1) is non-Lipschitz, the existing results on the CLT of the numerical temporal average are not applicable to (1.1). In view of the above consideration, we are devoted to investigating the CLT of the numerical temporal average for general SODEs with non-Lipschitz coefficients.

In this work, we consider the following SODE:

$$\mathrm{d}X(t) = b\big(X(t)\big)\mathrm{d}t + \sigma\big(X(t)\big)\mathrm{d}W(t), \quad t > 0, \tag{1.2}$$

where W is the same one defined in (1.1). Here, $b : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the strong dissipation condition and is allowed to grow super-linearly, and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times D}$ is bounded and Lipschitz continuous (see Section 2 for the detailed assumptions on b and σ). Then, (1.2) admits a unique strong solution on $[0, +\infty)$ for any given deterministic initial value $X(0) \in \mathbb{R}^d$. It is shown in [20, Theorem 3.1] that (1.2) admits a unique invariant measure π and is thus ergodic, due to the strong dissipation condition on b. Our main purpose is to study the CLT of the temporal average of the backward Euler-Maruyama (BEM) method applied to (1.2). The reasons for the choice of the BEM method are as follows:

(1) The CLT of the numerical temporal average characterizes the long-time behavior of numerical solutions. Thus, one preference in the choice of the numerical method is that it should possess the long-time stability. The Euler–Maruyama (EM) method and the BEM method are used most frequently when simulating SODEs (see e.g. [10, p. 453]). Similar to the deterministic case, the BEM method shows a more excellent long-time stability than the EM method.

(2) When applied to SODEs with super-linearly growing coefficients, the EM method is known to diverge [15]. Other explicit numerical methods based on the Itô-Taylor expansion could suffer from the same fate, for SODEs with super-linearly growing coefficients [10]. Thus, as is pointed out by [21], for SODEs with super-linearly growing coefficients, one usually adopts implicit numerical methods or modified explicit numerical methods such as the adaptive time step size method, tamed method and the truncated method.