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A HIGH ORDER SCHEME FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH THE CAPUTO-HADAMARD DERIVATIVE*

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Abstract

In this paper, we consider numerical solutions of the fractional diffusion equation with the α order time fractional derivative defined in the Caputo-Hadamard sense. A high order time-stepping scheme is constructed, analyzed, and numerically validated. The contribution of the paper is twofold: 1) regularity of the solution to the underlying equation is investigated, 2) a rigorous stability and convergence analysis for the proposed scheme is performed, which shows that the proposed scheme is $3+\alpha$ order accurate. Several numerical examples are provided to verify the theoretical statement.

Mathematics subject classification: 34A08, 65L05, 65L20.
Key words: Caputo-Hadamard derivative, Fractional differential equations, High order scheme, Stability and convergence analysis.

1. Introduction

Fractional calculus has been paid much attention in recent decades, due to, on one side, its well-recognized applicability in science and engineering, and on the other side, its attractive complementary of the integer order calculus in pure mathematics, see, e.g. [7, 19, 23, 24] and the references cited therein.

Up to now, there exist several kinds of fractional integrals and derivatives, like Riemann-Liouville, Caputo, Riesz, C-Fabrizio and Hadamard integrals and derivatives. The first three have been widely studied in the past decades. Actually, the Hadamard derivative which was proposed early in 1892 [15] is also very worthy of in-depth study, since it has been extensively used in mechanics and engineering, e.g. both planar and three-dimensional elasticities, or the fracture analysis [2] and the Lomnitz logarithmic creep law of special substances, e.g. igneous rock [11, 22]. Moreover, ultraslow diffusion appears in various applications [6, 9]. For instance, vacancy-mediated tracer flow and particle movements in certain strongly heterogeneous media may demonstrate ultraslow diffusive phenomena [4, 5, 28]. Mathematically, the mean square displacement of the particles in ultraslow diffusion grows logarithmically in time [16, 25, 26].

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Therefore, the Hadamard fractional operators, whose kernels are defined in terms of logarithmic functions, serve as a natural choice for modeling ultraslow diffusion processes and thus attract wide attentions.

For partial differential equations involving Hadamard derivative, although the research is relatively sparse, several studies have been carried out, and we see increasing interest in this topic from both scientific and engineering communities. We mention, among others, the work [18] to develop fractional integration and differentiation in the Hadamard setting. The existence almost everywhere was established for the considered Hadamard-type fractional derivative, the semigroup and reciprocal properties for the Hadamard-type fractional derivative and integration operators were proved. The stability and logarithmic decay of the solution of Hadamard-type fractional differential equation was discussed in [20]. A logarithmic transformation reducing the Caputo-Hadamard (CH) fractional problems to their Caputo analogues was presented in [29]. The well-posedness and regularity of CH fractional stochastic differential equations were studied in [28]. Numerically, Gohar *et al.* [12, 13] and Li *et al.* [21] derived several finite difference schemes to approximate the CH fractional derivative. Very recently, Fan *et al.* [10] derived some new numerical formulas, called as L1-2 formula, L2-1σ formula and H2N2 formula, for discretization of the CH fractional derivative. A second-order scheme with nonuniform time meshes for CH fractional sub-diffusion equations with initial singularity is investigated in [27]. The predictor-corrector numerical method for solving CH fractional differential equations with the graded meshes was considered in [14]. However, to the best of our knowledge, the convergence order of the existing schemes is no more than three.

The aim of this work is to propose and analyze an efficient time stepping scheme having the convergence order more than three for the CH fractional differential equations. The proposed scheme is based on the so-called block-by-block approach, which is a common method for the integral equations, and has been successfully applied to construct high order scheme for the Caputo fractional differential equations in [3]. Although the used idea for the scheme construction is the same as [3], the convergence analysis is a completely different skill from the method used in [3].

The rest of this paper is organized as follows. In Section 2, we present some regularity properties of the solution for the considered problem. In Section 3, we describe the detailed construction of the high order scheme for the Hadamard FDEs under consideration. Then in Section 4, we derive an estimate for the local errors through a series of lemmas. The stability and convergence analysis is given in Section 5. Finally, several numerical examples are provided in Section 6 to support the theoretical statement. Some concluding remarks are given in the final section.

2. Problem and Regularity Properties

We are interested in the following CH fractional equation with $0 < \alpha < 1$:

$$\begin{aligned} &{}^{CH}D_{a,t}^{\alpha} u(t) = f\big(t, u(t)\big), \quad 0 < a < t, \\ &u(a) = u_a, \end{aligned} \tag{2.1}$$

where $f(t, u)$ is a nonlinear function with respect to u, and the initial value u_a is given. The notation ${}^{CH}D_{a,t}^{\alpha}$ is the CH fractional derivative of order α defined by [2, 17],

$${}^{CH}D_{a}^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} \delta v(s) \frac{\mathrm{d}s}{s},$$