Journal of Computational Mathematics
Vol.43, No.5, 2025, 1063–1091.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2508-m2025-0035

EFFICIENT ACCELERATION STRATEGIES FOR MULTIGRID PRECONDITIONED CONJUGATE GRADIENTS IN FAST 3D TOPOLOGY OPTIMIZATION*

Bingzhen Zhou, Zixian Zhu and Xiaoping Wang[1)]
School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen 518172, China; Shenzhen International Center for Industrial and Applied Mathematics, Shenzhen Research Institute of Big Data, Shenzhen 518172, China
Emails: zbzhen@hunnu.edu.cn, zixianzhu@link.cuhk.edu.cn, wangxiaoping@cuhk.edu.cn

Abstract

This paper presents various acceleration techniques tailored for the traditional 3D topology optimization problem. Firstly, the adoption of the finite difference method leads to a sparser stiffness matrix, resulting in more efficient matrix-vector multiplication. Additionally, a fully matrix-free technique is proposed, which only assembles stiffness matrices at the coarsest grid level and does not require complex node numbering. Moreover, an innovative N-cycle multigrid (MG) algorithm is proposed to act as a preconditioner within conjugate gradient (CG) iterations. Finally, to further enhance the optimization process on high-resolution grids, a progressive strategy is implemented. The numerical results confirm that these acceleration techniques are not only efficient, but also capable of achieving lower compliance and reducing memory consumption. MATLAB codes complementing the article can be downloaded from Github.

Mathematics subject classification: 49M41, 65M55, 74B05, 74S20, 90C06.
Key words: Topology optimization, Linear elasticity, Fully matrix-free, Multigrid preconditioned conjugate gradient, Finite difference method.

1. Introduction

Structural topology optimization (TO) is a mathematical method used for designing the layout of materials or structures. It aims to find the most efficient material distribution within a given design space, considering load conditions and constraints. This method is used in various engineering fields, such as aerospace, automotive, mechanical engineering, architecture [6,9,27], heat sinks [23,33], and flow networks [16].

The origins of TO trace back to Michell's work in 1904, which derived a solution method for classic Michell truss structures [20]. Over time, various methods have been proposed to establish a stable and efficient numerical algorithm for solving TO, which are applicable in engineering practice. Depending on the description of the density function, methods such as the solid isotropic material with penalization (SIMP), level set methods [29], isogeometric analysis [31], moving morphable components [36], and phase field techniques [11] have emerged. Among these methods, SIMP is widely adopted. This method translates the topology optimization

* Received January 26, 2025 / Revised version received June 17, 2025 / Accepted August 13, 2025 / Published online September 9, 2025 /
[1)] Corresponding author

problem into a nonlinear programming problem with multiple variables, thereby avoiding the need for methods that require second-order derivatives. Commonly used algorithms for solving TO include the optimality criteria method [6, 38], method of moving asymptotes (MMA) [28], sequential linear programming [26], and threshold dynamic method [8, 15, 16].

Topology optimization has seen rapid development over the past two decades, numerous review articles have emerged [6, 9, 17, 27, 30, 35]. However, it has surprisingly not yet gained mainstream acceptance among structural engineers. A key challenge in topology optimization is handling large-scale or high-dimensional design problems that may involve millions to billions of degrees of freedom [2]. The most expensive step in topology optimization algorithms is the computation of the objective function, which involves solving a linear system at each iteration. High-performance computing approaches have been successful in addressing these challenges [1, 2, 37]; however, they require substantial computational resources and do not reduce computational demands. Strategies have been applied to reduce the cost of computation, such as reanalysis [25], reduced-order models [12, 32], and multigrid methods [3, 4, 24].

Various acceleration strategies for large-scale problems are discussed in [13, 21, 22, 31, 32, 34]. In this paper, we employ a simple two-stage approach [18], where the optimal results from the first stage with a coarse mesh is used as the initial design for the second stage with a finer mesh. The optimized result from the first stage can accelerate the convergence of the second stage. By recursively applying the two-stage approach, the main optimization process occurs on the coarse mesh. Performing most of the optimization on the coarse grid and only a few optimization loops on the fine grid significantly enhances efficiency.

This paper aims to provide a robust and efficient algorithm for solving 3D problems by integrating existing technologies with new strategies thereby achieving higher quality solutions. Our goals include running 3D topology optimization problems efficiently on personal computers and promoting integration with computer-aided design (CAD) software. The various acceleration strategies proposed in this paper are based on the multigrid preconditioned conjugate gradient (MGCG) method. A new N-cycle algorithm, similar to the V-cycle and W-cycle multigrid methods, offers the acceleration benefits of the CG method. The use of the finite difference method, as opposed to finite elements, accelerates matrix-vector multiplication and results in lower compliance. The numerical equivalence between finite difference and finite element methods has been established, which allows for straightforward algorithm implementation by simply replacing the stiffness matrix. One of the most efficient acceleration techniques is the fully matrix-free approach, which aims to reduce storage requirements and boost computational efficiency. As this approach obviates the need to assemble large-scale matrices, it enables high-resolution problems to be executed on personal computers.

The remainder of this paper is organized as follows. Section 2 introduces the 3D linear elasticity problem and the topology optimization problem. Section 3 details various acceleration strategies. Section 4 verifies the effectiveness of these acceleration strategies with numerical examples.

2. Model Problem

2.1. Linear elasticity problem

The classic 3D linear elasticity problem can be formulated as the following equilibrium equation with boundary conditions: