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SHARP ERROR ESTIMATE OF VARIABLE TIME-STEP IMEX BDF2 SCHEME FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH INITIAL SINGULARITY ARISING IN FINANCE*

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Abstract

The recently developed DOC kernels technique has been successful in the stability and convergence analysis for variable time-step BDF2 schemes. However, it may not be readily applicable to problems exhibiting an initial singularity. In the numerical simulations of solutions with initial singularity, variable time-step schemes like the graded mesh are always adopted to achieve the optimal convergence, whose first adjacent time-step ratio may become pretty large so that the acquired restriction is not satisfied. In this paper, we revisit the variable time-step implicit-explicit two-step backward differentiation formula (IMEX BDF2) scheme to solve the parabolic integro-differential equations (PIDEs) with initial singularity. We obtain the sharp error estimate under a mild restriction condition of adjacent time-step ratios $r_k := \tau_k/\tau_{k-1} < r_{\max} = 4.8645$ $(k \geq 3)$ and a much mild requirement on the first ratio, i.e. $r_2 > 0$. This leads to the validation of our analysis of the variable time-step IMEX BDF2 scheme when the initial singularity is dealt by a simple strategy, i.e. the graded mesh $t_k = T(k/N)^\gamma$. In this situation, the convergence order of $\mathcal{O}(N^{-\min\{2,\gamma\alpha\}})$ is achieved, where N denotes the total number of mesh points and α indicates the regularity of the exact solution. This is, the optimal convergence will be achieved by taking $\gamma_{\rm opt} = 2/\alpha$. Numerical examples are provided to demonstrate our theoretical analysis.

Mathematics subject classification: 65M06, 65M12.
Key words: Implicit-explicit method, Two-step backward differentiation formula, The discrete orthogonal convolution kernels, The discrete complementary convolution kernels, Error estimates, Variable time-step.

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1. Introduction

In this paper, we consider the computation of parabolic integro-differential equations, which arise in option pricing theory when the underlying asset follows a jump diffusion process [2,12, 25,34,37]

$$\begin{aligned} &\partial_t u(x,t) - c_1 u_{xx}(x,t) + c_2 u_x(x,t) + c_3 u + \mathcal{J}\big(u(x,t)\big) = f(x,t), && (x,t) \in \Omega \times (0,T], \\ &u(x,t) = u_b(x,t), && (x,t) \in \partial\Omega \times (0,T], \\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega = (x_l, x_r)$ with its boundary $\partial\Omega$, the parameters $c_1 > 0$, $c_2, c_3 \in \mathbb{R}$. Here $\mathcal{J}(\cdot)$ represents the nonlocal integral operator and is defined by

$$\mathcal{J}(u) := \int_\Omega u(z,t)\rho(x-z)\,\mathrm{d}z,$$

where $\rho : \mathbb{R} \to \mathbb{R}^+$ is a given function satisfying $\|\rho(x)\|_{L^\infty} \le C$ for some positive constant C. In this situation, there exists a constant $\hat{C}_\rho$ (only depends on the given function ρ and Ω) such that

$$\|\mathcal{J}(u)\|_{L^2(\Omega)} \le \hat{C}_\rho \|u\|_{L^2(\Omega)}. \tag{1.2}$$

The bound (1.2) holds in numerous practical problems, including the finite activity jump diffusion model (e.g. Merton model and Kou model), CGMY model and KoBoL with infinite activity and finite variation [6,31].

There are two main considerations in numerically solving PIDEs (1.1). The first one involves the discretization of nonlocal integral operators. Undue discretization of the nonlocal integral operator with finite difference schemes, such as the fully implicit time stepping scheme, may increase the computational complexity since inversions will be required to solve the resulting systems with full matrices at each time step. To deal with the full matrix, variable methods are designed such as iterative methods [1,9,24,32], FFT [2,9] and the alternating direction implicit (ADI) method [2]. An alternative approach is to avoid the inversion of a full matrix. To this end, an increasingly popular alternative is the implicit-explicit (IMEX) method [12,14,15,27,34], which typically treats the nonlocal integral term explicitly and the rest part implicitly. The IMEX method leads to a tridiagonal system and can be solved efficiently.

A feature of PIDEs (1.1) is the weak singularity near the initial time $t = 0$ arising from nonsmooth initial data. For example, as noted in [34], with a nonsmooth payoff function in option models, the regularity of the exact solution $u(x,t)$ may exhibit the form

$$\left\|\partial_t^k u\right\|_{L^2} \le C t^{\frac{1}{2}-k}, \quad k = 1,2,3.$$

To be more general, in this paper, we consider the regularity assumption as follows.

Assumption 1.1. There exists a constant $\bar{C}$ such that the solution to (1.1) satisfies

$$\left\|\partial_t^k u\right\| \le \bar{C} t^{-k+\alpha}, \quad t \in (0,T], \quad k = 1,2,3, \tag{1.3}$$

where $\partial_t^k u := \partial^k u / \partial t^k$ and the regularity parameter α satisfies $1/2 \le \alpha < 1$.

Therefore, another consideration is how to develop efficient and accurate algorithms for solving problems involving initial singularities. A heuristic method to improve efficiency without