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TRUNCATED EULER-MARUYAMA METHOD FOR TIME-CHANGED STOCHASTIC DIFFERENTIAL EQUATIONS WITH SUPER-LINEAR STATE VARIABLES AND HÖLDER'S CONTINUOUS TIME VARIABLES*

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Abstract

An explicit numerical method is developed for a class of non-autonomous time-changed stochastic differential equations, whose coefficients obey Hölder's continuity in terms of the time variables and are allowed to grow super-linearly in terms of the state variables. The strong convergence of the method in the finite time interval is proved and the convergence rate is obtained. Numerical simulations are provided.

Mathematics subject classification: 65C30, 60H10.
Key words: Explicit numerical method, Highly non-linear coefficients, Time-changed processes, Stochastic differential equations, Strong convergence.

1. Introduction

Time-changed processes and time-changed stochastic differential equations (SDEs), as important mathematical tools to describe subdiffusions, have been broadly investigated in recent decades [1, 17, 20, 22, 28, 29, 34]. As explicit forms of solutions to time-changed SDEs are rarely obtained, numerical methods become extremely important when this type of SDEs are applied in practice.

For time-changed SDEs of the following form:

$$dY(t) = f\big(E(t), Y(t)\big)dE(t) + g\big(E(t), Y(t)\big)dB\big(E(t)\big), \tag{1.1}$$

the Euler-Maruyama (EM) method was proposed in [15], in which both the coefficients f and g were required to satisfy the global Lipschitz condition in terms of the state variable. To our best knowledge, [15] is the first paper that investigates the finite time convergence of the numerical method for time-changed SDEs. When the constraint on f is released to the one-sided Lipschitz condition for (1.1), the semi-implicit EM method was studied in [3]. Furthermore, the truncated EM method was discussed in [21] when some super-linear terms are allowed to appear in both the coefficients f and g. All the three works mentioned above employed the duality principle that was developed in [17]. Briefly speaking, the duality principle states that the solution to (1.1) can be represented as $X(E(t))$, where $X(t)$ is the solution to the classical SDE

$$dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dB(t). \tag{1.2}$$

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Therefore, one can construct numerical methods for (1.1) by combining a numerical method to (1.2) with the discretization of $E(t)$.

However, such a one-to-one duality principle does not exist for many other forms of time-changed SDEs. Thus, discretizing the time-changed SDE directly cannot be avoided for more general time-changed SDEs. For example, the time-changed SDEs of the form

$$dY(t) = f\big(t, Y(t)\big)dE(t) + g\big(t, Y(t)\big)dB\big(E(t)\big) \tag{1.3}$$

do not have the dual classical SDEs. Therefore, the EM method was proposed in [14] by the direct discretization of the equation, where the assumption imposed on both f and g is the global Lipschitz condition in terms of the state variable. Such a requirement excludes many time-changed SDEs, like

$$dY(t) = \big(Y(t) - 5Y^3(t)\big)dE(t) + Y^2(t)dB\big(E(t)\big), \tag{1.4}$$

where some super-linear terms appear in both f and g. In the case of the classical SDE, i.e. $E(t)$ is replaced by t in (1.4), it was proved that the classical EM method fails to converge to such equations [10]. There are many alternative methods have been proposed to handle the super-linearity, for example, the semi-implicit EM method [8, 9, 27], the tamed-type methods [4, 11, 32, 33, 35], the truncated-type methods [2, 18, 19, 26, 36], and the adaptive time-stepping methods [5,12,16,30]. We just mention some of works here and refer the readers to the references therein.

Although there is no existing work like [10] that directly proves the divergence of the EM method for the time-changed SDEs with super-linear growing coefficients, it is not hard to see by using the similar arguments in [10] that the EM method is not convergent for this type of time-changed SDEs. To be slightly more precise, this is because that the time-changed Brownian motion is an unbounded stochastic process and would lead to the divergence of the EM method just like what the Brownian motion did for the EM method in the case of the classical SDEs. Therefore, some alternatives are needed to deal with time-changed SDEs like (1.4).

In this paper, we adopt the truncating idea to develop the truncated EM method for a class of highly non-linear time-changed SDEs. It should be pointed out that the truncating function employed in this work may not be applicable for the numerical reproduction of long-time behaviours like stabilities or invariant measures of the underlying equations, but adopting other truncating functions could enable our method suitable for numerical studies of asymptotic properties of the underlying equations [19]. Due to the length of this paper, we focus on the finite time strong convergence in this works and may investigate other properties in some future works.

The main contributions of this work are as follows:

- Strong convergence of the proposed method is proved when the state variables satisfy the polynomial growth condition and the time variables obey the Hölder continuity condition.
- The convergence rate is obtained to be $\min\{\gamma_f, \gamma_g, 1/2 - \varepsilon\}$ for arbitrarily small $\varepsilon > 0$, where γ_f and γ_g are the Hölder index for the time variables.

The rest of this paper is arranged in the following way. The mathematical preliminaries are put in Section 2. The main results and proofs are presented in Section 3. Numerical examples that illustrate the theoretical results are provided in Section 4. Section 5 concludes the paper and discusses some future research.