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LOGISTIC STOCHASTIC DIFFERENTIAL EQUATIONS WITH POWER-LAW*

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Abstract

An analysis of logistic stochastic differential equations (SDEs) with general power-law and driven by a Wiener process is conducted. We prove existence of unique, strong Markovian, continuous solutions. The solutions live (a.s.) on bounded domains $D = [0, K]$ required by applications to biology, ecology and physics with nonrandom threshold parameter $K > 0$ (i.e. the maximum carrying constant). Moreover, we present and justify nonstandard numerical methods constructed by specified balanced implicit methods (BIMs). Their weak and L^p-convergence follows from the fact that these methods with local Lipschitz-continuous coefficients of logistic SDEs "produce" positive numerical approximations on bounded domain $[0, K]$ (a.s.). As commonly known, standard numerical methods such as Taylor-type ones for SDEs fail to do that. Finally, asymptotic stability of nontrivial equilibria $x_* = K$ is proven for both continuous time logistic SDEs and discrete time approximations by BIMs. We exploit the technique of positive, sufficiently smooth and Lyapunov functionals governed by well-known Dynkin's formula for SDEs.

Mathematics subject classification: 34F05, 60H10, 60H35, 65C30, 92-10, 92D25, 92D40.
Key words: Logistic stochastic differential equations, Existence of bounded unique solutions, Asymptotic stability, positivity, Convergence, Balanced implicit methods.

1. Introduction

Consider Itô-type stochastic differential equations

$$dX(t) = \lambda \cdot X \cdot \left(1 - \left[\frac{X}{K}\right]^{\gamma}\right) dt + \sigma X \cdot g(X)\, dW(t), \tag{1.1}$$

where $\lambda, K, \gamma, \sigma > 0$ are positive real constants and $W = (W(t))_{t \geq 0}$ forms a standard Wiener process. $g : \mathbb{R}^1 \to \mathbb{R}^1$ is supposed to be a local Lipschitz-continuous function on $(0, K)$ with $g(K) = 0$. Such equations occur in mathematical biology to model logistic phenomena (such as random SI models in epidemics with $\gamma = 1$, e.g. see [6, 15, 19]), marketing and social sciences (random Bass model with $\gamma = 1$, e.g. [15, 16]) or in field theory of mathematical physics (a randomized version of Ginzburg equation with $\gamma = 2$, e.g. [8, 19]). These models (1.1) are useful to model the offspring-to-parents ratio

$$\frac{dX}{X} \approx a(X)\, dt + b(X)\, dW(t)$$

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by semimartingales which are locally Gaussian distributed under the condition $X(t) = x$, motivated by the central role of Gaussian distributions supported by the well-known central limit theorem (CLT). In any case, random fluctuations such as determined by Wiener process W have to be taken into account due to erratic behavior of dynamically evolving data in physics, engineering, economy, finance and sciences. So, we are motivated to study their basic properties, a.s. boundedness of solutions, asymptotic stability of their equilibria and justify qualitatively adequate discretizations of such models (1.1) from mathematical point of view (i.e. both in discrete time and continuous time).

We do not know the analytic form of their solutions for most of those models (1.1). Hence, we need to resort to numerical approximations. There are plenty of numerical methods for SDEs. However, they still experience a lack of justification by qualitative features which are present in adequate, biologicially and physicially relevant data. Stability, positivity and boundedness are such properties, whereas much more is known about quantitative statements such as convergence rates of those numerical methods with Lipschitz-continuous coefficient functions. In [19] we studied the special case of $\gamma = 1$. Here we shall concentrate on a more thorough study of more general case of $\gamma > 0$ under $g(K) = 0$. The aim is to derive and justify models with solutions which exclusively stay on $[0, K]$ ($\mathbb{P}$-a.s.). This aim stems from the fact that biologically relevant models for population dynamics must reside within that range and naturally maximum, finite thresholds K exists in physically relevant data. In contrast to that fact, most of the authors on stochastic biological models consider only the restriction of positive populations. Standard numerical methods such as of Taylor-type have not been proved to stay on compact sets such as intervals $[0, K]$ (a.s.). To the best of our knowledge, positivity of stochastic numerical methods is considered in [15] at first. There the class of balanced implicit methods is successfully exploited. We shall present numerical realizations of logistic SDEs (1.1) with powers $\gamma > 0$ through BIMs guaranteeing (a.s.) to stay on compact sets $[0, K]$ as well, despite unbounded random fluctuations inherited from the noise of underlying SDEs.

For the general theory related to SDEs, see books of [1,3–7,10–14]. For a basic introduction to stochastic calculus and SDEs, see [9,21]. It is worth noting that, as one of the first, Gard [6] has realized the importance of continuous time SDEs for modeling biologically relevant dynamics of large populations. Later this work applied to populations has significantly been continued and specified by Allen [2] using Markov chains.

This paper is organized as follows. Section 2 presents an existence and uniqueness result for continuous time, Markovian, strong solutions of model (1.1) which exclusively reside in intervals $\mathbb{D} = [0, K]$. Section 3 studies stochastic, almost sure and moment asymptotic stability of equilibria. $\mathbb{D}$-invariant numerical methods (i.e. easily implementable, balanced implicit methods without any direct taming of noise terms) are verified by theorems and lemmas on their a.s. boundedness, L^2- and weak convergence rates in Section 4. Section 5 proves asymptotic stochastic stability of saturation equilibrium $x_* = K$ of BIMs under appropriate conditions on possibly variable, but nonrandom step sizes (note that the problem of algorithms with random step sizes and nonlinear implicitness is the loss of predictability/measurability of numerical approximations with respect to basic requirements of Itô calculus! This problem can be by-passed by the use of linear-implicit numerical methods such as BIMs). Eventually, Section 6 reports on some simple numerical experiments with 2D surface plots of expected Lyapunov functionals $\mathbb{E}[V]$ depending on diverse parameters and supporting our findings. A short Section 7 with a summary, remarks and further suggestions concludes this paper.